

# EXISTENCE AND PROPERTIES OF SEMI-BOUNDED GLOBAL SOLUTIONS TO THE FUNCTIONAL DIFFERENTIAL EQUATION WITH VOLTERRA'S TYPE OPERATORS ON THE REAL LINE

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ABSTRACT. Consider the equation

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + f(u)(t) \quad \text{for a. e. } t \in \mathbb{R}$$

where  $\ell_i : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  ( $i = 0, 1$ ) are linear positive continuous operators and  $f : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  is a continuous operator satisfying the local Carathéodory conditions. The efficient conditions guaranteeing the existence of a global solution, which is bounded and non-negative in the neighbourhood of  $-\infty$ , to the equation considered are established provided  $\ell_0$ ,  $\ell_1$ , and  $f$  are Volterra's type operators. The existence of a solution which is positive on the whole real line is discussed, as well. Furthermore, the asymptotic properties of such solutions are studied in the neighbourhood of  $-\infty$ . The results are applied to certain models appearing in natural sciences.

## 1. INTRODUCTION

Many models in natural sciences are based on the idea that the derivative at a certain moment of time depends not only on the present state but on some of the previous states. However, in spite of the fact that the history of delay differential equations goes back to the beginning of the 20th century (see, e.g., the works of Vito Volterra), or even more back in time, the systematic study of such types of equations started only in the beginning of the 1950s.

The main purpose of the present paper is to study the existence and asymptotic properties of a global solution (i.e., defined on the whole real line) to the scalar functional differential equation

$$(1.1) \quad u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + f(u)(t).$$

Here,  $\ell_i : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  ( $i = 0, 1$ ) are linear continuous operators which are positive, i.e., they transform non-negative functions into the set of non-negative functions, and  $f : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  is a continuous operator satisfying the local Carathéodory conditions, i.e., for every  $r > 0$  there exists  $q_r \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  such that

$$|f(v)(t)| \leq q_r(t) \quad \text{for a. e. } t \in \mathbb{R}$$

whenever

$$\sup \{|v(t)| : t \in \mathbb{R}\} \leq r.$$

Together with the equation (1.1) consider the condition

$$(1.2) \quad u(t_0) = c$$

with  $t_0, c \in \mathbb{R}$ .

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By a global solution to the equation (1.1) we understand a function  $u : \mathbb{R} \rightarrow \mathbb{R}$  which is absolutely continuous on every compact interval and satisfies (1.1) for almost every  $t \in \mathbb{R}$ . Effective sufficient conditions for the existence of a global solution to the problem (1.1), (1.2) are established in the paper. More precisely, we are interested in the study of existence of global positive semi-bounded (i.e., bounded in the neighbourhood of  $-\infty$ ) solutions  $u : \mathbb{R} \rightarrow \mathbb{R}$  to the problem (1.1), (1.2).

The study of the geometric property and the existence of solutions to this class of problems was motivated by the open problem concerned with degenerate scalar reaction-diffusion equations with delay

$$\phi_t(t, x) = \phi_{xx}(t, x) - \phi(t, x) + G(\phi(t - r, x)), \quad x \in \mathbb{R}, \quad r > 0,$$

and the existence of positive semi-wavefront solutions  $\phi(t, x) = u(x + ct)$ ,  $u(-\infty) = 0$ , when  $G \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ ,  $G'(0) = 1$ , and 0 and  $\kappa > 0$  are the only two solutions of  $G(s) = s$  (degenerate monostable case). When we do not consider diffusion, we obtain the following equation with the boundary condition

$$(1.3) \quad \begin{cases} cu'(t) = -u(t) + G(u(t - cr)), \\ u(-\infty) = 0, \end{cases}$$

with degenerate monostable nonlinearity  $G$ . The existence problem for (1.3) and their generalizations have been investigated in several papers and approached by means of different methods and almost always assuming the generate condition  $G'(0) > 1$ . It is worthwhile mentioning that in the proofs of existence this condition is essential and cannot be omitted or weakened within the framework (see [2, 6, 9] and references therein).

In the case when  $r = 0$ , without delay, only a few theoretical studies have considered the important problem when  $G'(0) = 1$ , i.e., the degenerate case (see [3, 10]). These works show that the assumption  $G'(0) > 1$  is not necessary to obtain the existence and the geometric properties of travelling solutions of a nonlocal dispersal problem or parabolic equations. Motivated for these investigations we have developed a more general theory that can be applied to the problem with delay (1.3) and hence to complete or to improve the research on existence problems done so far.

The results of the paper can be also applied to the scalar delay logistic equation of the form

$$(1.4) \quad u'(t) = u(t)F(t, u_t)$$

describing the population growth (see Section 6). The asymptotic properties at  $+\infty$  of such kinds of models where studied e.g. in [1, 5] (see also references therein). For more model differential equations used in natural sciences which can be rewritten in the form of (1.1) we recommend [1] and references therein.

In this way, the main results of our work can be reformulated for a particular case of the equation (1.1), for the equation with argument deviation of the form

$$(1.5) \quad u'(t) = p_0(t)u(\mu_0(t)) - p_1(t)u(\mu_1(t)) + h(t, u(t), u(\nu(t))),$$

where  $p_i \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$ ,  $\mu_i, \nu : \mathbb{R} \rightarrow \mathbb{R}$  are locally essentially bounded measurable functions,  $\mu_i(t) \leq t$ ,  $\nu(t) \leq t$  for almost every  $t \in \mathbb{R}$  ( $i = 0, 1$ ), and  $h : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a function satisfying local Carathéodory conditions, i.e.,  $h(\cdot, x, y) : \mathbb{R} \rightarrow \mathbb{R}$  is measurable for every  $x, y \in \mathbb{R}$ ,  $h(t, \cdot, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous for almost every  $t \in \mathbb{R}$ , and for every  $r > 0$  there exists  $q_r \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  such that

$$|h(t, x, y)| \leq q_r(t) \quad \text{for a. e. } t \in \mathbb{R}, \quad |x| + |y| \leq r.$$

Note that, when  $p_0(t) = p_1(t) = 1/c$ ,  $\mu_0(t) = \nu(t) = t - cr$ ,  $\mu_1(t) = t$  and  $h(t, u(t), u(\nu(t))) = [G(u(t - cr)) - u(t - cr)]/c$ , equation (1.5) reduces to the model equation (1.3).

The paper is organized as follows. Basic notation used in the paper can be found just after this introduction in Section 1. The main results dealing with the existence and positivity of a global semi-bounded solution to the problem (1.1), (1.2), as well as the conditions guaranteeing that such a solution has a limit at  $-\infty$  equal to zero, are established in Section 2. The results of Section 2 are reformulated for the particular case of (1.1)—the equation (1.5)—in Section 3. Sections 4 and 5 are devoted to the auxiliary propositions and proofs of the main results, respectively. Applications of the obtained results to the model problem (1.3) and generalized logistic equation (1.4) can be found in Section 6.

**1.1. Basic notation.** The following notation is used throughout the paper:

$\mathbb{N}$  is a set of all natural numbers.

$\mathbb{R}$  is a set of all real numbers;  $\mathbb{R}_+ = [0, +\infty)$ ;  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ ;  $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ .

$C([a, b]; \mathbb{R})$  is a Banach space of continuous functions  $u : [a, b] \rightarrow \mathbb{R}$  with the norm

$$\|u\|_{a,b} = \max \{|u(t)| : t \in [a, b]\}.$$

$C([a, b]; \mathbb{R}_+) = \{u \in C([a, b]; \mathbb{R}) : u(t) \geq 0 \text{ for } t \in [a, b]\}.$

$AC([a, b]; D)$ , where  $D \subseteq \mathbb{R}$ , is a set of absolutely continuous functions  $u : [a, b] \rightarrow D$ .

$L([a, b]; \mathbb{R}_+)$  is a set of Lebesgue-integrable functions  $p : [a, b] \rightarrow \mathbb{R}_+.$

$C_{loc}(\mathbb{R}; \mathbb{R})$  is a space of continuous functions  $u : \mathbb{R} \rightarrow \mathbb{R}$  with the topology of uniform convergence on every compact interval.

If  $u \in C_{loc}(\mathbb{R}; \mathbb{R})$  then  $u(-\infty)$ , resp.  $u(+\infty)$ , stands for a limit (finite or infinite) of  $u$  at  $-\infty$ , resp.  $+\infty$ , if such a limit exists.

$C_0(\mathbb{R}; \mathbb{R})$  is a Banach space of bounded continuous functions  $u : \mathbb{R} \rightarrow \mathbb{R}$  with the norm

$$\|u\| = \sup \{|u(t)| : t \in \mathbb{R}\}.$$

$C_0(I; D)$ , where  $I \subseteq \mathbb{R}$ ,  $D \subseteq \mathbb{R}$ , is a set of bounded continuous functions  $u : I \rightarrow D$ .

$AC_{loc}(I; D)$ , where  $I \subseteq \mathbb{R}$ ,  $D \subseteq \mathbb{R}$ , is a set of functions  $u : I \rightarrow D$  which are absolutely continuous on every compact interval contained in  $I$ .

$L_{loc}(\mathbb{R}; \mathbb{R})$  is a space of locally Lebesgue-integrable functions  $p : \mathbb{R} \rightarrow \mathbb{R}$  with the topology of convergence in the mean on every compact interval.

$L_{loc}(I; \mathbb{R}_+)$ , where  $I \subseteq \mathbb{R}$ , is a set of functions  $p : I \rightarrow \mathbb{R}_+$  which are Lebesgue-integrable on every compact interval contained in  $I$ .

$L^{+\infty}(\mathbb{R}; \mathbb{R})$  is a Banach space of essentially bounded measurable functions  $p : \mathbb{R} \rightarrow \mathbb{R}$  with the norm

$$\|p\|_\infty = \text{ess sup} \{|p(t)| : t \in \mathbb{R}\}.$$

$K([a, b] \times \mathbb{R}_+; \mathbb{R}_+)$  is the Carathéodory class, i.e., the set of functions  $q : [a, b] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $q(\cdot, x) : [a, b] \rightarrow \mathbb{R}_+$  is measurable for any  $x \in \mathbb{R}_+$ ,  $q(t, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous for almost all  $t \in [a, b]$ , and

$$\sup \{q(\cdot, x) : x \in D\} \in L([a, b]; \mathbb{R}_+)$$

for any compact set  $D \subset \mathbb{R}_+.$

$K_{loc}(I \times \mathbb{R}_+; \mathbb{R}_+)$ , where  $I \subseteq \mathbb{R}$ , is a set of functions  $q : I \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $q \in K([a, b] \times \mathbb{R}_+; \mathbb{R}_+)$  for every  $[a, b] \subset I$ .

Let  $I \subseteq \mathbb{R}$  be a closed interval,  $T : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  be a continuous operator, and let  $u : I \rightarrow \mathbb{R}$  be a continuous function. Then we put

$$T(u)(t) \stackrel{\text{def}}{=} T(\vartheta(u))(t) \quad \text{for a. e. } t \in \mathbb{R}$$

where

$$(1.6) \quad \vartheta(u)(t) = \begin{cases} u(\inf I) & \text{for } t < \inf I \text{ if } \inf I > -\infty, \\ u(t) & \text{for } t \in I, \\ u(\sup I) & \text{for } t > \sup I \text{ if } \sup I < +\infty. \end{cases}$$

$\mathcal{P}_\tau^+$ , where  $\tau \in \mathbb{R}$ , is a set of all linear continuous operators  $\ell : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  such that

$$\ell(u)(t) \geq 0 \quad \text{for a. e. } t \leq \tau$$

whenever  $u \in AC_{loc}((-\infty, \tau]; \mathbb{R}_+)$  is a nondecreasing function.

$V_\tau$ , where  $\tau \in \mathbb{R}$ , is a set of all continuous operators  $T : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  such that, for arbitrary  $\zeta \leq \tau$ , the equality

$$T(u)(t) = T(v)(t) \quad \text{for a. e. } t \leq \zeta$$

holds whenever  $u, v \in C_{loc}(\mathbb{R}; \mathbb{R})$  are such that

$$u(t) = v(t) \quad \text{for } t \leq \zeta.$$

$\Sigma$  is a set of all continuous functions  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $\sigma(t) \leq t$  for  $t \in \mathbb{R}$ .

$V_\tau(\sigma)$ , where  $\tau \in \mathbb{R}$  and  $\sigma \in \Sigma$ , is a set of all continuous operators  $T : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  with a memory  $\sigma$  on  $(-\infty, \tau]$ , i.e., for almost every  $t \leq \tau$ , the equality

$$T(u)(t) = T(v)(t)$$

holds provided  $u, v \in C_{loc}(\mathbb{R}; \mathbb{R})$  are such that

$$u(s) = v(s) \quad \text{for } s \in [\sigma(t), t].$$

$$V = \bigcap_{\tau \in \mathbb{R}} V_\tau, \quad \mathcal{P}^+ = \bigcap_{\tau \in \mathbb{R}} \mathcal{P}_\tau^+.$$

Let  $I \subseteq \mathbb{R}$  be a closed interval. By a solution to the equation (1.1) on the interval  $I$  we understand a function  $u : I \rightarrow \mathbb{R}$  which is absolutely continuous on every compact interval contained in  $I$  and satisfies (1.1) almost everywhere on  $I$ .<sup>1</sup> If, moreover,  $t_0 \in I$  then by a solution to the problem (1.1), (1.2) on  $I$  we understand a solution  $u$  to (1.1) on  $I$  satisfying (1.2). If  $I = \mathbb{R}$  then we speak about a global solution.

## 2. MAIN RESULTS

### 2.1. Existence theorems.

**Theorem 2.1.** *Let  $\ell_0, \ell_1, f \in V$ ,  $\sigma \in \Sigma$ ,*

$$(2.1) \quad \ell_0 \in V_{t_0}(\sigma),$$

$$(2.2) \quad f(0)(t) = 0 \quad \text{for a. e. } t \leq t_0,$$

*and let there exist  $\kappa > 0$  such that*

$$(2.3) \quad f(v)(t) \geq 0 \quad \text{for a. e. } t \leq t_0, \quad v \in C_0(\mathbb{R}; [0, \kappa])$$

*and*

$$(2.4) \quad f(v)(t) \operatorname{sgn} v(t) \leq q(t, \|v\|) \quad \text{for a. e. } t \geq t_0, \quad v \in C_0(\mathbb{R}; \mathbb{R}),$$

$$0 \leq v(t_0) \leq \kappa, \quad -\kappa \leq v(t) \leq \kappa \quad \text{for } t \leq t_0,$$

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<sup>1</sup>Remind that by  $\ell_0(u)$ ,  $\ell_1(u)$ , and  $f(u)$  we understand  $\ell_0(\vartheta(u))$ ,  $\ell_1(\vartheta(u))$ , and  $f(\vartheta(u))$ , respectively, where  $\vartheta$  is given by (1.6).

where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  is nondecreasing in the second argument and satisfies

$$(2.5) \quad \lim_{x \rightarrow +\infty} \frac{1}{x} \int_{t_0}^b q(s, x) ds = 0$$

for every  $b > t_0$ . Let, moreover, there exist  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  such that

$$(2.6) \quad \gamma'(t) \leq -\ell_1(\gamma)(t) \quad \text{for a. e. } t \leq t_0,$$

$$(2.7) \quad \ell_0(1)(t) \geq \frac{\ell_1(\gamma)(t)}{\gamma(t)} \quad \text{for a. e. } t \leq t_0,$$

$$(2.8) \quad \sup \left\{ \int_{\sigma(t)}^t \frac{\ell_1(\gamma)(s)}{\gamma(s)} ds : t \leq t_0 \right\} < +\infty.$$

Then, for every  $c \in [0, \kappa e^{-M_\sigma}]$ , where

$$(2.9) \quad M_\sigma = \sup \left\{ \int_{\sigma(t)}^t \frac{\ell_1(\gamma)(s)}{\gamma(s)} ds : t \leq t_0 \right\},$$

there exists a global solution  $u$  to the problem (1.1), (1.2) satisfying

$$(2.10) \quad 0 \leq u(t) \leq \kappa \quad \text{for } t \leq t_0.$$

**Remark 2.1.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 2.1, has also a finite limit  $u(-\infty)$  according to Theorem 2.8 formulated below.

**Theorem 2.2.** Let  $\ell_0, \ell_1, f \in V$  be such that  $\ell_0 - \ell_1 \in \mathcal{P}_{t_0}^+$ , (2.2) holds, and let there exist  $\kappa > 0$  such that (2.3) and (2.4) are fulfilled, where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  is nondecreasing in the second argument and satisfies (2.5) for every  $b > t_0$ . Then, for every  $c \in [0, \kappa]$  there exists a global solution  $u$  to the problem (1.1), (1.2) satisfying (2.10) and

$$(2.11) \quad u'(t) \geq 0 \quad \text{for a. e. } t \leq t_0.$$

**Remark 2.2.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 2.2, has also a finite limit  $u(-\infty)$  because  $u$  is a bounded nondecreasing function in the neighbourhood of  $-\infty$ .

**Theorem 2.3.** Let all the assumptions of Theorem 2.1 be fulfilled. Let, moreover, there exist  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  and a continuous nondecreasing function  $h_0 : (0, +\infty) \rightarrow (0, +\infty)$  such that

$$(2.12) \quad f(v)(t) \leq g(t)h_0(\|v\|) \quad \text{for a. e. } t \leq t_0, \quad v \in C_0(\mathbb{R}; [0, \kappa]), \quad v \not\equiv 0$$

and

$$(2.13) \quad \lim_{x \rightarrow 0^+} \int_x^1 \frac{ds}{h_0(s)} = +\infty.$$

Then, for every  $c \in (0, \kappa e^{-M_\sigma}]$  with  $M_\sigma$  given by (2.9), there exists a global solution  $u$  to the problem (1.1), (1.2) satisfying

$$(2.14) \quad 0 < u(t) \leq \kappa \quad \text{for } t \leq t_0.$$

**Remark 2.3.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 2.3, has also a finite limit  $u(-\infty)$  according to Theorem 2.8 formulated below.

**Theorem 2.4.** Let all the assumptions of Theorem 2.2 be fulfilled. Let, moreover, there exist  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  and a continuous nondecreasing function  $h_0 : (0, +\infty) \rightarrow (0, +\infty)$  such that (2.12) and (2.13) hold. Then, for every  $c \in (0, \kappa]$  there exists a global solution  $u$  to the problem (1.1), (1.2) satisfying (2.11) and (2.14).

**Remark 2.4.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 2.4, has also a finite limit  $u(-\infty)$  because  $u$  is a bounded nondecreasing function in the neighbourhood of  $-\infty$ .

**Theorem 2.5.** *Let  $\ell_0, \ell_1, f \in V$ ,  $\sigma \in \Sigma$ , (2.1) and (2.2) hold, and let there exist  $\kappa > 0$  such that (2.3) and (2.4) are fulfilled where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  is nondecreasing in the second argument and satisfies (2.5) for every  $b > t_0$ . Let, moreover,*

$$(2.15) \quad \ell_0(v)(t) + f(v)(t) \geq 0 \quad \text{for a. e. } t \geq t_0, \quad v \in C_0(\mathbb{R}; \mathbb{R}_+), \quad v(t) \leq \kappa \quad \text{for } t \leq t_0.$$

*Let, in addition, there exist  $\gamma \in AC_{loc}(\mathbb{R}; (0, +\infty))$  such that*

$$(2.16) \quad \gamma'(t) \leq -\ell_1(\gamma)(t) \quad \text{for a. e. } t \in \mathbb{R},$$

*and (2.7) and (2.8) are satisfied. Then, for every  $c \in (0, \kappa e^{-M_\sigma}]$ , where  $M_\sigma$  is given by (2.9), there exists a global solution  $u$  to the problem (1.1), (1.2) satisfying (2.10) and*

$$(2.17) \quad u(t) > 0 \quad \text{for } t \geq t_0.$$

**Remark 2.5.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 2.5, has also a finite limit  $u(-\infty)$  according to Theorem 2.8 formulated below.

**Theorem 2.6.** *Let  $\ell_0, \ell_1, f \in V$  be such that  $\ell_0 - \ell_1 \in \mathcal{P}^+$ , (2.2) holds, and let there exist  $\kappa > 0$  such that (2.3) and*

$$(2.18) \quad 0 \leq f(v)(t) \leq q(t, \|v\|) \quad \text{for a. e. } t \geq t_0, \quad v \in C_0(\mathbb{R}; \mathbb{R}_+), \quad v(t) \leq \kappa \quad \text{for } t \leq t_0$$

*are fulfilled, where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  is nondecreasing in the second argument and satisfies (2.5) for every  $b > t_0$ . Then, for every  $c \in [0, \kappa]$  there exists a global solution  $u$  to the problem (1.1), (1.2) satisfying (2.10) and*

$$(2.19) \quad u'(t) \geq 0 \quad \text{for a. e. } t \in \mathbb{R}.$$

**Remark 2.6.** Obviously, if  $c > 0$  in Theorem 2.6 then (2.19) implies (2.17).

**Remark 2.7.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 2.6, has also a finite limit  $u(-\infty)$  because  $u$  is a bounded nondecreasing function in the neighbourhood of  $-\infty$ .

Theorems 2.3–2.6 together with Remark 2.6 imply the following results dealing with the existence of global solutions to (1.1), (1.2) which are positive on the whole real line.

**Corollary 2.1.** *Let all the assumptions of Theorem 2.5 be fulfilled. Let, moreover, there exist  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  and a continuous nondecreasing function  $h_0 : (0, +\infty) \rightarrow (0, +\infty)$  such that (2.12) and (2.13) hold. Then, for every  $c \in (0, \kappa e^{-M_\sigma}]$  with  $M_\sigma$  given by (2.9), there exists a positive global solution  $u$  to the problem (1.1), (1.2) satisfying*

$$(2.20) \quad u(t) \leq \kappa \quad \text{for } t \leq t_0.$$

**Corollary 2.2.** *Let all the assumptions of Theorem 2.6 be fulfilled. Let, moreover, there exist  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  and a continuous nondecreasing function  $h_0 : (0, +\infty) \rightarrow (0, +\infty)$  such that (2.12) and (2.13) hold. Then, for every  $c \in (0, \kappa]$  there exists a positive global solution  $u$  to the problem (1.1), (1.2) satisfying (2.19) and (2.20).*

**Remark 2.8.** Note that, according to Remarks 2.5 and 2.7, the solution  $u$ , the existence of which is guaranteed by Corollary 2.1, resp. Corollary 2.2, has also a finite limit  $u(-\infty)$ .

Corollaries 2.1 and 2.2 are direct consequences of Theorems 2.3–2.6 and Remark 2.6. Therefore, their proofs are omitted.

## 2.2. Properties of solutions.

**Theorem 2.7.** *Let  $\ell_1 \in V_{t_0}$  and let there exist  $\kappa > 0$  such that*

$$(2.21) \quad \ell_0(v)(t) + f(v)(t) \geq 0 \quad \text{for a. e. } t \leq t_0, \quad v \in C_0(\mathbb{R}; [0, \kappa]).$$

*Let, moreover, there exist  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  such that (2.6) is fulfilled and*

$$(2.22) \quad \gamma(-\infty) < +\infty.$$

*Then every solution  $u$  to (1.1) on  $(-\infty, t_0]$  satisfying (2.10) has a finite limit  $u(-\infty)$ .*

**Remark 2.9.** Note that the conditions (2.6), (2.22), and the inclusion  $\ell_1 \in V_{t_0}$  imply that the function  $\ell_1(1)$  is integrable in the neighbourhood of  $-\infty$ , i.e.,

$$(2.23) \quad \lim_{t \rightarrow -\infty} \int_t^{t_0} \ell_1(1)(s) ds < +\infty.$$

Indeed, from (2.6) it follows that  $\gamma$  is a nonincreasing function, and thus

$$(2.24) \quad \gamma'(t) \leq -\ell_1(1)(t)\gamma(t) \quad \text{for a. e. } t \leq t_0$$

(see Lemma 4.1 below with  $\ell = \ell_1$ ,  $\alpha = 1$ ,  $\beta = -\vartheta(\gamma)$  and  $\vartheta$  given by (1.6)). Now (2.24) yields

$$(2.25) \quad 0 < \gamma(t_0) \leq \gamma(t) \exp \left( - \int_t^{t_0} \ell_1(1)(s) ds \right) \quad \text{for } t \leq t_0$$

and so (2.23) holds provided (2.22) is fulfilled.

On the other hand, from (2.25) it follows that if

$$\lim_{t \rightarrow -\infty} \int_t^{t_0} \ell_1(1)(s) ds = +\infty,$$

then necessarily  $\gamma(-\infty) = +\infty$ . If the latter occurs, one can apply the following theorem.

**Theorem 2.8.** *Let  $\ell_1 \in V_{t_0}$ ,  $\sigma \in \Sigma$ , (2.1) hold, and let there exist  $\kappa > 0$  such that (2.3) is fulfilled. Let, moreover, there exist  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  such that (2.6)–(2.8) are satisfied. Then every solution  $u$  to (1.1) on  $(-\infty, t_0]$  satisfying (2.10) has a finite limit  $u(-\infty)$ .*

To formulate our next result we introduce

**Definition 2.1.** Let  $\omega \in \Sigma$ , and let  $\tau \in \mathbb{R}$ ,  $\kappa > 0$ , and  $c \in (0, \kappa)$  be constants. An operator  $T : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  belongs to the set  $\mathcal{O}_\tau(\omega, \kappa, c)$  if

$$\limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t T(u)(s) ds > 0$$

whenever  $u \in C_0(\mathbb{R}; [0, \kappa])$  is such that

$$u(\tau) = c,$$

there exists a finite limit  $u(-\infty) \leq c$ , and

$$\begin{aligned} 0 < u(-\infty) \leq u(t) & \quad \text{for } t \leq \tau \quad \text{if } u(-\infty) < c, \\ u(t) = c & \quad \text{for } t \leq \tau \quad \text{if } u(-\infty) = c. \end{aligned}$$

**Theorem 2.9.** *Let  $\ell_0, \ell_1 \in V_{t_0}$ ,*

$$(2.26) \quad \ell_0(1)(t) \geq \ell_1(1)(t) \quad \text{for a. e. } t \leq t_0,$$

*and let there exist  $\kappa > 0$  such that (2.3) holds. Let, moreover, there exist a function  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  such that (2.6) and (2.22) are fulfilled. Assume, further, that  $c \in (0, \kappa)$*

and  $u$  is a solution to the problem (1.1), (1.2) on  $(-\infty, t_0]$  satisfying (2.10) and having a limit  $u(-\infty)$ . If either

$$(2.27) \quad \lim_{t \rightarrow -\infty} \int_t^{t_0} \ell_0(1)(s) ds = +\infty$$

or there exists  $\omega \in \Sigma$  such that

$$(2.28) \quad f \in \mathcal{O}_{t_0}(\omega, \kappa, c)$$

then

$$(2.29) \quad u(-\infty) = 0.$$

**Theorem 2.10.** *Let  $\ell_0, \ell_1 \in V_{t_0}$ , (2.26) hold, and let there exist  $\kappa > 0$  such that (2.3) is satisfied. Let, moreover, there exist  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  satisfying (2.6). Assume, further, that  $c \in (0, \kappa)$  and  $u$  is a solution to the problem (1.1), (1.2) on  $(-\infty, t_0]$  satisfying (2.10) and having a limit  $u(-\infty)$ . If there exists  $\omega \in \Sigma$  such that*

$$(2.30) \quad \sup \left\{ \int_{\omega(t)}^t \ell_1(1)(s) ds : t \leq t_0 \right\} < +\infty,$$

and either

$$(2.31) \quad \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t [\ell_0(1)(s) - \ell_1(1)(s)] ds > 0$$

or (2.28) holds then (2.29) is fulfilled.

Now we formulate a sufficient condition for the inclusion (2.28).

**Proposition 2.1.** *Let  $f \in V_{t_0}$ ,  $\omega \in \Sigma$ , and let there exist  $\kappa > 0$ ,  $g \in L_{loc}(\mathbb{R}; \mathbb{R})$ , and a continuous operator  $h_1 : C_0(\mathbb{R}; \mathbb{R}) \rightarrow L^{+\infty}(\mathbb{R}; \mathbb{R})$  such that*

$$(2.32) \quad f(v)(t) \geq g(t)h_1(v)(t) \quad \text{for a. e. } t \leq t_0, \quad v \in C_0(\mathbb{R}; [0, \kappa]).$$

Let, moreover,

$$(2.33) \quad \sup \left\{ \int_{\omega(t)}^t |g(s)| ds : t \leq t_0 \right\} < +\infty$$

and

$$(2.34) \quad \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t g(s)h_1(x)(s) ds > 0$$

for every constant function  $x : \mathbb{R} \rightarrow (0, \kappa)$ . Then (2.28) holds for every  $c \in (0, \kappa)$ .

**Corollary 2.3.** *Let  $\ell_0, \ell_1, f \in V_{t_0}$ , (2.26) hold, and let there exist  $\kappa > 0$ ,  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$ , and a continuous operator  $h_1 : C_0(\mathbb{R}; \mathbb{R}) \rightarrow L^{+\infty}(\mathbb{R}; \mathbb{R})$  such that*

$$(2.35) \quad h_1(v)(t) \geq 0 \quad \text{for a. e. } t \leq t_0, \quad v \in C_0(\mathbb{R}; [0, \kappa])$$

and (2.32) is satisfied. Let, moreover, there exist  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  satisfying (2.6) and (2.22). Assume, further, that  $c \in (0, \kappa)$  and either (2.27) holds or

$$(2.36) \quad \lim_{t \rightarrow -\infty} \int_t^{t_0} g(s) ds = +\infty,$$

$$(2.37) \quad \lim_{t \rightarrow -\infty} \text{ess inf} \{ h_1(x)(s) : s \leq t \} > 0$$



for every constant function  $x : \mathbb{R} \rightarrow (0, \kappa)$ . Then every solution  $u$  to the problem (1.1), (1.2) on  $(-\infty, t_0]$  satisfying (2.10) has a finite limit  $u(-\infty)$  and (2.29) holds.

**Corollary 2.4.** *Let  $\ell_1, f \in V_{t_0}$ ,  $\sigma \in \Sigma$ , (2.1) hold, and let there exist  $\kappa > 0$ ,  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$ , and a continuous operator  $h_1 : C_0(\mathbb{R}; \mathbb{R}) \rightarrow L^{+\infty}(\mathbb{R}; \mathbb{R})$  such that (2.32) and (2.35) are satisfied. Let, moreover, there exist  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  satisfying (2.6)–(2.8). Assume, further, that  $c \in (0, \kappa)$  and there exists  $\omega \in \Sigma$  such that (2.30) is fulfilled and either (2.31) holds or (2.33) and (2.34) for every constant function  $x : \mathbb{R} \rightarrow (0, \kappa)$  are satisfied. Then every solution  $u$  to the problem (1.1), (1.2) on  $(-\infty, t_0]$  satisfying (2.10) has a finite limit  $u(-\infty)$  and (2.29) holds.*

### 3. EQUATION WITH DEVIATING ARGUMENTS

Now we establish assertions dealing with the equation (1.5).

#### 3.1. Existence theorems.

**Theorem 3.1.** *Let there exist  $\kappa > 0$  such that*

$$(3.1) \quad h(t, x, y) \geq 0 \quad \text{for a. e. } t \leq t_0, \quad x, y \in [0, \kappa],$$

$$(3.2) \quad h(t, x, y) \operatorname{sgn} x \leq q(t, |x| + |y|) \quad \text{for a. e. } t \geq t_0, \quad x, y \in \mathbb{R},$$

where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  is nondecreasing in the second argument and satisfies (2.5) for every  $b > t_0$ . Let, moreover,

$$(3.3) \quad h(t, 0, 0) = 0 \quad \text{for a. e. } t \leq t_0,$$

$$(3.4) \quad \mu_0(t) \leq t, \quad \mu_1(t) \leq t, \quad \nu(t) \leq t \quad \text{for a. e. } t \in \mathbb{R},$$

$$(3.5) \quad \int_{\mu_1(t)}^t p_1(s) ds \leq \frac{1}{e} \quad \text{for a. e. } t \leq t_0,$$

$$(3.6) \quad p_0(t) \geq p_1(t) \exp \left( e \int_{\mu_1(t)}^t p_1(s) ds \right) \quad \text{for a. e. } t \leq t_0,$$

$$(3.7) \quad \operatorname{ess\,sup} \left\{ \int_{\mu_0(t)}^t p_1(s) ds : t \leq t_0 \right\} < +\infty.$$

Then, for every  $c \in [0, \kappa e^{-M_\mu})$ , where

$$(3.8) \quad M_\mu = \operatorname{ess\,sup} \left\{ \int_{\mu_0(t)}^t p_1(s) \exp \left( e \int_{\mu_1(s)}^s p_1(\xi) d\xi \right) ds : t \leq t_0 \right\},$$

there exists a global solution  $u$  to the problem (1.5), (1.2) satisfying (2.10).

**Remark 3.1.** Condition (3.6) in Theorem 3.1 can be weakened to

$$(3.9) \quad p_0(t) \geq p_1(t) \exp \left( \lambda \int_{\mu_1(t)}^t p_1(s) ds \right) \quad \text{for a. e. } t \leq t_0$$

where  $\lambda \in [1, e]$  satisfies

$$(3.10) \quad \lambda = e^{\lambda p^*}, \quad p^* = \operatorname{ess\,sup} \left\{ \int_{\mu_1(t)}^t p_1(s) ds : t \leq t_0 \right\}.$$

Obviously, in that case the number  $M_\mu$  can also be improved in an appropriate sense.

**Remark 3.2.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 3.1, has also a finite limit  $u(-\infty)$  (see Theorem 3.8).

**Theorem 3.2.** *Let there exist  $\kappa > 0$  such that (3.1) and (3.2) hold, where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  is nondecreasing in the second argument and satisfies (2.5) for every  $b > t_0$ . Let, moreover, (3.3) and (3.4) be fulfilled. Assume further that*

$$(3.11) \quad p_0(t) \geq p_1(t) \quad \text{for a. e. } t \leq t_0,$$

$$(3.12) \quad p_1(t)(\mu_0(t) - \mu_1(t)) \geq 0 \quad \text{for a. e. } t \leq t_0.$$

*Then, for every  $c \in [0, \kappa]$  there exists a global solution  $u$  to the problem (1.5), (1.2) satisfying (2.10) and (2.11).*

**Remark 3.3.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 3.2, has also a finite limit  $u(-\infty)$  because  $u$  is a bounded nondecreasing function.

**Theorem 3.3.** *Let all the assumptions of Theorem 3.1 be fulfilled. Let, moreover, there exist  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  and a continuous nondecreasing function  $h_0 : (0, +\infty) \rightarrow (0, +\infty)$  such that*

$$(3.13) \quad h(t, x, y) \leq g(t)h_0(x + y) \quad \text{for a. e. } t \leq t_0, \quad x, y \in \mathbb{R}_+, \quad x + y \neq 0$$

*and (2.13) holds. Then, for every  $c \in (0, \kappa e^{-M_\mu})$  with  $M_\mu$  given by (3.8), there exists a global solution  $u$  to the problem (1.5), (1.2) satisfying (2.14).*

**Remark 3.4.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 3.3, has also a finite limit  $u(-\infty)$  (see Theorem 3.8).

**Theorem 3.4.** *Let all the assumptions of Theorem 3.2 be fulfilled. Let, moreover, there exist  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  and a continuous nondecreasing function  $h_0 : (0, +\infty) \rightarrow (0, +\infty)$  such that (2.13) and (3.13) hold. Then, for every  $c \in (0, \kappa]$  there exists a global solution  $u$  to the problem (1.5), (1.2) satisfying (2.11) and (2.14).*

**Remark 3.5.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 3.4, has also a finite limit  $u(-\infty)$  because  $u$  is a bounded nondecreasing function.

**Theorem 3.5.** *Let there exist  $\kappa > 0$  such that (3.1) and (3.2) hold where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  is nondecreasing in the second argument and satisfies (2.5) for every  $b > t_0$ . Let, moreover, (3.3), (3.4), (3.6), (3.7), and*

$$(3.14) \quad \int_{\mu_1(t)}^t p_1(s) ds \leq \frac{1}{e} \quad \text{for a. e. } t \in \mathbb{R}$$

*be fulfilled. Let, in addition,*

$$(3.15) \quad h(t, x, y) \geq 0 \quad \text{for a. e. } t \geq t_0, \quad x, y \in \mathbb{R}_+.$$

*Then, for every  $c \in (0, \kappa e^{-M_\mu})$ , where  $M_\mu$  is given by (3.8), there exists a global solution  $u$  to the problem (1.5), (1.2) satisfying (2.10) and (2.17).*

**Remark 3.6.** Note that, according to Theorem 2.5 (see also the proof of Theorem 3.5), in the case when  $\mu_0(t) = t$  for almost every  $t \geq t_0$ , resp.  $\mu_0(t) = \nu(t)$  for almost every  $t \geq t_0$ , the condition (3.15) in Theorem 3.5 can be weakened to

$$p_0(t)x + h(t, x, y) \geq 0 \quad \text{for a. e. } t \geq t_0, \quad x, y \in \mathbb{R}_+,$$

resp.

$$p_0(t)y + h(t, x, y) \geq 0 \quad \text{for a. e. } t \geq t_0, \quad x, y \in \mathbb{R}_+.$$

Moreover, the condition (3.6) in Theorem 3.5 can be weakened to (3.9) where  $\lambda \in [1, e]$  satisfies (3.10). Obviously, in that case the number  $M_\mu$  can also be improved in an appropriate sense.

**Remark 3.7.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 3.5, has also a finite limit  $u(-\infty)$  (see Theorem 3.8).

**Theorem 3.6.** *Let there exist  $\kappa > 0$  such that (3.1) holds and let*

$$(3.16) \quad 0 \leq h(t, x, y) \leq q(t, x + y) \quad \text{for a. e. } t \geq t_0, \quad x, y \in \mathbb{R}_+$$

*where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  is nondecreasing in the second argument and satisfies (2.5) for every  $b > t_0$ . Let, moreover, (3.3) and (3.4) be fulfilled. Assume further that*

$$(3.17) \quad p_0(t) \geq p_1(t) \quad \text{for a. e. } t \in \mathbb{R},$$

$$(3.18) \quad p_1(t)(\mu_0(t) - \mu_1(t)) \geq 0 \quad \text{for a. e. } t \in \mathbb{R}.$$

*Then, for every  $c \in [0, \kappa]$  there exists a global solution  $u$  to the problem (1.5), (1.2) satisfying (2.10) and (2.19).*

**Remark 3.8.** Obviously, if  $c > 0$  in Theorem 3.6 then (2.19) implies (2.17).

**Remark 3.9.** Note that the solution  $u$ , the existence of which is guaranteed by Theorem 3.6, has also a finite limit  $u(-\infty)$  because  $u$  is a bounded nondecreasing function.

Theorems 3.3–3.6 together with Remark 3.8 imply the following results dealing with the existence of global solutions to (1.5), (1.2) which are positive on the whole real line.

**Corollary 3.1.** *Let all the assumptions of Theorem 3.5 be fulfilled. Let, moreover, there exist  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  and a continuous nondecreasing function  $h_0 : (0, +\infty) \rightarrow (0, +\infty)$  such that (2.13) and (3.13) hold. Then, for every  $c \in (0, \kappa e^{-M_\mu})$  with  $M_\mu$  given by (3.8), there exists a positive global solution  $u$  to the problem (1.5), (1.2) satisfying (2.20).*

**Corollary 3.2.** *Let all the assumptions of Theorem 3.6 be fulfilled. Let, moreover, there exist  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  and a continuous nondecreasing function  $h_0 : (0, +\infty) \rightarrow (0, +\infty)$  such that (2.13) and (3.13) hold. Then, for every  $c \in (0, \kappa]$  there exists a positive global solution  $u$  to the problem (1.5), (1.2) satisfying (2.19) and (2.20).*

**Remark 3.10.** Note that, according to Remarks 3.7 and 3.9, the solution  $u$ , the existence of which is guaranteed by Corollary 3.1, resp. Corollary 3.2, has also a finite limit  $u(-\infty)$ .

Corollaries 3.1 and 3.2 are direct consequences of Theorems 3.3–3.6 and Remark 3.8. Therefore, their proofs are omitted.

### 3.2. Properties of solutions.

**Theorem 3.7.** *Let there exist  $\kappa > 0$  such that (3.1) holds. Let, moreover,*

$$(3.19) \quad \mu_0(t) \leq t, \quad \mu_1(t) \leq t, \quad \nu(t) \leq t \quad \text{for a. e. } t \leq t_0,$$

$$(3.20) \quad \lim_{t \rightarrow -\infty} \int_t^{t_0} p_1(s) ds < +\infty,$$

*and (3.5) be fulfilled. Then every solution  $u$  to (1.5) on  $(-\infty, t_0]$  satisfying (2.10) has a finite limit  $u(-\infty)$ .*

**Theorem 3.8.** *Let there exist  $\kappa > 0$  such that (3.1) holds. Let, moreover, (3.5)–(3.7) and (3.19) be fulfilled. Then every solution  $u$  to (1.5) on  $(-\infty, t_0]$  satisfying (2.10) has a finite limit  $u(-\infty)$ .*

**Theorem 3.9.** *Let there exist  $\kappa > 0$ ,  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$ , and a continuous function  $h_1 : (0, \kappa) \times (0, \kappa) \rightarrow \mathbb{R}$  such that*

$$(3.21) \quad h_1(x, x) > 0 \quad \text{for } x \in (0, \kappa),$$

$$(3.22) \quad h(t, x, y) \geq g(t)h_1(x, y) \quad \text{for a. e. } t \leq t_0, \quad x, y \in (0, \kappa).$$

*Let, moreover, (3.19) and (3.20) be fulfilled. Assume, further, that  $u$  is a solution to (1.5) on  $(-\infty, t_0]$  having a limit  $u(-\infty) \in [0, \kappa]$ . If either*

$$(3.23) \quad \lim_{t \rightarrow -\infty} \int_t^{t_0} p_0(s) ds = +\infty$$

*or (3.26) holds then either (2.29) is fulfilled or*

$$(3.24) \quad u(-\infty) = \kappa.$$

**Theorem 3.10.** *Let there exist  $\kappa > 0$ ,  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$ , and a continuous function  $h_1 : (0, \kappa) \times (0, \kappa) \rightarrow \mathbb{R}$  such that (3.21) and (3.22) hold. Let, moreover, (3.5), (3.11), and (3.19) be fulfilled. Assume, further, that  $u$  is a solution to (1.5) on  $(-\infty, t_0]$  having a limit  $u(-\infty) \in [0, \kappa]$ . If there exists  $\omega \in \Sigma$  such that*

$$(3.25) \quad \sup \left\{ \int_{\omega(t)}^t p_1(s) ds : t \leq t_0 \right\} < +\infty,$$

*and either*

$$(3.26) \quad \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t [p_0(s) - p_1(s)] ds > 0$$

*or*

$$(3.27) \quad \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t g(s) ds > 0$$

*then either (2.29) or (3.24) holds.*

**Corollary 3.3.** *Let there exist  $\kappa > 0$ ,  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$ , and a continuous function  $h_1 : (0, \kappa) \times (0, \kappa) \rightarrow \mathbb{R}_+$  such that (3.21) and (3.22) hold. Let, moreover, (3.5), (3.19), and (3.20) be fulfilled. Assume, further, that either (2.36) or (3.23) is satisfied. Then every solution  $u$  to (1.5) on  $(-\infty, t_0]$  satisfying (2.10) has a finite limit  $u(-\infty)$  and either (2.29) or (3.24) holds.*

**Corollary 3.4.** *Let all the assumptions of Corollary 3.3 be fulfilled. If, in addition, (3.11) holds, then every solution  $u$  to (1.5) on  $(-\infty, t_0]$  satisfying (2.10) has a finite limit  $u(-\infty)$  and either (2.29) holds or*

$$(3.28) \quad u(t) = \kappa \quad \text{for } t \leq t_0.$$

**Corollary 3.5.** *Let there exist  $\kappa > 0$ ,  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$ , and a continuous function  $h_1 : (0, \kappa) \times (0, \kappa) \rightarrow \mathbb{R}_+$  such that (3.21) and (3.22) hold. Let, moreover, (3.5)–(3.7) and (3.19) be fulfilled. Assume, further, that*

$$\sup \left\{ \int_{t-1}^t p_1(s) ds : t \leq t_0 \right\} < +\infty,$$

*and either*

$$\limsup_{t \rightarrow -\infty} \int_{t-1}^t [p_0(s) - p_1(s)] ds > 0$$

*or*

$$\limsup_{t \rightarrow -\infty} \int_{t-1}^t g(s) ds > 0.$$

Then every solution  $u$  to (1.5) on  $(-\infty, t_0]$  satisfying (2.10) has a finite limit  $u(-\infty)$  and either (2.29) or (3.28) holds.

**Remark 3.11.** Note that (3.28) can be fulfilled only if

$$\begin{aligned} p_0(t) &= p_1(t) & \text{for a. e. } t \leq t_0, & \quad h(t, \kappa, \kappa) = 0 & \text{for a. e. } t \leq t_0, \\ & \int_{\mu_1(t)}^t p_1(s) ds = 0 & \text{for a. e. } t \leq t_0 \end{aligned}$$

provided all the assumptions of Corollary 3.4 or Corollary 3.5 are fulfilled.

#### 4. AUXILIARY PROPOSITIONS

**4.1. Preliminaries.** First we introduce some already known results which will be used later.

**Definition 4.1.** Let  $a, b \in \mathbb{R}$ ,  $a < b$ . A linear continuous operator  $\ell : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  is said to belong to the set  $\mathcal{S}_{ab}(a)$ , resp.  $\mathcal{S}_{ab}(b)$ , if every function  $u \in AC([a, b]; \mathbb{R})$  satisfying

$$(4.1) \quad u'(t) \geq \ell(u)(t) \quad \text{for a. e. } t \in [a, b], \quad u(a) \geq 0,$$

resp.

$$u'(t) \leq \ell(u)(t) \quad \text{for a. e. } t \in [a, b], \quad u(b) \geq 0,$$

admits the inequality

$$(4.2) \quad u(t) \geq 0 \quad \text{for } t \in [a, b].$$

**Definition 4.2.** Let  $a, b \in \mathbb{R}$ ,  $a < b$ . A linear continuous operator  $\ell : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  is said to belong to the set  $\mathcal{S}'_{ab}(a)$  if every function  $u \in AC([a, b]; \mathbb{R})$  satisfying (4.1) admits the inequalities (4.2) and

$$u'(t) \geq 0 \quad \text{for a. e. } t \in [a, b].$$

**Proposition 4.1** (see [7, Corollary 1.1]). Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_0 \in V_{t_0}$ . Then  $\ell_0 \in \mathcal{S}_{at_0}(a)$ .

**Proposition 4.2** (see [7, Theorem 1.2]). Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_1 \in V_{t_0}$ , and let there exist a function  $\gamma \in AC([a, t_0]; (0, +\infty))$  such that

$$(4.3) \quad \gamma'(t) \leq -\ell_1(\gamma)(t) \quad \text{for a. e. } t \in [a, t_0].$$

Then  $-\ell_1 \in \mathcal{S}_{at_0}(a)$ .

**Proposition 4.3** (see [7, Theorem 1.4]). Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_0 \in \mathcal{S}_{at_0}(a)$ ,  $-\ell_1 \in \mathcal{S}_{at_0}(a)$ . Then

$$(4.4) \quad \ell_0 - \ell_1 \in \mathcal{S}_{at_0}(a).$$

**Proposition 4.4** (see [7, Theorem 1.5]). Let  $a \in \mathbb{R}$ ,  $a < t_0$ . Then  $-\ell_1 \in \mathcal{S}_{at_0}(t_0)$  if and only if there exists  $\gamma \in AC([a, t_0]; (0, +\infty))$  satisfying (4.3).

**Remark 4.1.** Note that, according to Proposition 4.4, we have  $-\ell_1 \in \mathcal{S}_{a\tau}(\tau)$  for every  $\tau \in (a, t_0)$  provided  $-\ell_1 \in \mathcal{S}_{at_0}(t_0) \cap V_{t_0}$ .

**Proposition 4.5** (see [4, Theorem 6.2]). Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_0 - \ell_1 \in \mathcal{P}_{t_0}^+$ ,  $\ell_0 \in \mathcal{S}_{at_0}(a)$ . Then the problem

$$(4.5) \quad u'(t) = \ell_0(u)(t) - \ell_1(u)(t), \quad u(t_0) = 0$$

has on  $[a, t_0]$  only the trivial solution.

**Proposition 4.6** (see [4, Theorem 4.1]). Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_0 - \ell_1 \in \mathcal{P}_{t_0}^+$ ,  $\ell_0 \in \mathcal{S}_{at_0}(a)$ . Then

$$(4.6) \quad \ell_0 - \ell_1 \in \mathcal{S}'_{at_0}(a).$$

**Lemma 4.1.** *Let  $\ell : C_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow L_{loc}(\mathbb{R}; \mathbb{R})$  be a linear positive<sup>2</sup> continuous operator,  $\ell \in V_{t_0}$ ,  $\alpha \in C_{loc}(\mathbb{R}; \mathbb{R})$  be a non-negative function, and let  $\beta \in C_{loc}(\mathbb{R}; \mathbb{R})$  be a nondecreasing function. Then*

$$(4.7) \quad \ell(\alpha\beta)(t) \leq \ell(\alpha)(t)\beta(t) \quad \text{for a. e. } t \leq t_0.$$

*Proof.* Let  $A$  be a set of those points  $t \in (-\infty, t_0]$  where the derivatives

$$\frac{d}{dt} \int_t^{t_0} \ell(\alpha\beta)(s)ds \quad \text{and} \quad \frac{d}{dt} \int_t^{t_0} \ell(\alpha)(s)ds$$

exist and are equal to  $\ell(\alpha\beta)(t)$  and  $\ell(\alpha)(t)$ , respectively. Let  $t \in A$  be arbitrary but fixed. According to the inclusion  $\ell \in V_{t_0}$  we have

$$(4.8) \quad \ell(\alpha\beta)(s) \leq \ell(\alpha)(s)\beta(t) \quad \text{for a. e. } s \leq t.$$

Consequently, from (4.8) it follows that

$$(4.9) \quad \frac{1}{h} \int_{t-h}^t \ell(\alpha\beta)(s)ds \leq \frac{\beta(t)}{h} \int_{t-h}^t \ell(\alpha)(s)ds \quad \text{for } h > 0.$$

Passing to the limit as  $h$  tends to zero in (4.9), we get

$$\ell(\alpha\beta)(t) \leq \ell(\alpha)(t)\beta(t).$$

Since  $t \in A$  was arbitrary, from the latter inequality it follows that (4.7) holds.  $\square$

## 4.2. Lemmas on a finite interval.

**Lemma 4.2.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_0, \ell_1 \in V_{t_0}$ , and let*

$$(4.10) \quad -\ell_1 \in \mathcal{S}_{at_0}(t_0).$$

*Then the problem (4.5) has on  $[a, t_0]$  only the trivial solution.*

*Proof.* First note that according to Propositions 4.1–4.4, in view of (4.10), we have (4.4). Let  $u$  be a solution to the problem (4.5) on  $[a, t_0]$ . Obviously, without loss of generality we can assume that

$$(4.11) \quad u(a) \geq 0.$$

According to (4.4), in view of (4.5) and (4.11), we have

$$(4.12) \quad u(t) \geq 0 \quad \text{for } t \in [a, t_0].$$

Therefore, from (4.5) it follows that

$$(4.13) \quad u'(t) \geq -\ell_1(u)(t) \quad \text{for a. e. } t \in [a, t_0], \quad u(t_0) = 0.$$

However, according to (4.10) and (4.13), we have

$$(4.14) \quad u(t) \leq 0 \quad \text{for } t \in [a, t_0].$$

Now (4.12) and (4.14) results in  $u \equiv 0$ .  $\square$

---

<sup>2</sup>It transforms non-negative functions into the set of non-negative functions.

**Lemma 4.3.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_1 \in V_{t_0}$ , and let (2.26) and (4.10) be fulfilled. Let, moreover,  $u \in AC([a, t_0]; \mathbb{R}_+)$  satisfy*

$$(4.15) \quad u'(t) \geq \ell_0(u)(t) - \ell_1(u)(t) \quad \text{for a. e. } t \in [a, t_0].$$

*Then*

$$(4.16) \quad u(a) = \min \{u(t) : t \in [a, t_0]\},$$

*and, in addition, if there exists  $\tau \in (a, t_0]$  such that  $u(\tau) = u(a)$ , then*

$$(4.17) \quad u(t) = u(a) \quad \text{for } t \in [a, \tau].$$

*Proof.* To prove lemma it is sufficient to show that whenever there exists  $\tau \in (a, t_0]$  such that

$$(4.18) \quad u(\tau) = \min \{u(t) : t \in [a, t_0]\}$$

then  $u$  satisfies (4.17), and so (4.16) holds necessarily. Therefore, let  $\tau \in (a, t_0]$  be arbitrary but fixed, such that (4.18) holds. Put

$$(4.19) \quad z(t) = u(t) - u(\tau) \quad \text{for } t \in [a, \tau].$$

Then, in view of (2.26), (4.15), (4.18), and (4.19), we have

$$(4.20) \quad z(t) \geq 0 \quad \text{for } t \in [a, \tau],$$

$$(4.21) \quad z'(t) \geq \ell_0(1)(t)u(\tau) - \ell_1(u)(t) \geq -\ell_1(z)(t) \quad \text{for a. e. } t \in [a, \tau],$$

$$(4.22) \quad z(\tau) = 0.$$

Moreover, according to Remark 4.1, the inclusion (4.10) implies

$$(4.23) \quad -\ell_1 \in \mathcal{S}_{a\tau}(\tau).$$

Therefore, from (4.21) and (4.22) we get

$$(4.24) \quad z(t) \leq 0 \quad \text{for } t \in [a, \tau].$$

Now (4.19), (4.20), and (4.24) implies (4.17). □

**Lemma 4.4.** *Let  $a, \tau \in \mathbb{R}$ ,  $a < \tau$ , and let there exist  $\gamma \in AC([a, \tau]; (0, +\infty))$  satisfying*

$$(4.25) \quad \gamma'(t) \leq -\ell_1(\gamma)(t) \quad \text{for a. e. } t \in [a, \tau].$$

*Let, moreover,  $u \in AC([a, \tau]; \mathbb{R})$  be such that*

$$(4.26) \quad \max \{u(t) : t \in [a, \tau]\} \geq 0,$$

$$(4.27) \quad u'(t) \geq -\ell_1(u)(t) \quad \text{for a. e. } t \in [a, \tau].$$

*Then*

$$(4.28) \quad \max \left\{ \frac{u(t)}{\gamma(t)} : t \in [a, \tau] \right\} = \frac{u(\tau)}{\gamma(\tau)}.$$

*Proof.* Put

$$(4.29) \quad \lambda = \max \left\{ \frac{u(t)}{\gamma(t)} : t \in [a, \tau] \right\}.$$

Then, according to (4.25)–(4.27) and (4.29) we have  $\lambda \geq 0$ ,

$$(4.30) \quad \lambda\gamma(t) - u(t) \geq 0 \quad \text{for } t \in [a, \tau],$$

$$(4.31) \quad \lambda\gamma'(t) - u'(t) \leq -\ell_1(\lambda\gamma - u)(t) \quad \text{for a. e. } t \in [a, \tau],$$

and there exists  $\tau_0 \in [a, \tau]$  such that

$$(4.32) \quad \lambda\gamma(\tau_0) - u(\tau_0) = 0.$$

However, from (4.30) and (4.31) it follows that  $\lambda\gamma - u$  is a nonincreasing function, which together with (4.29), (4.30), and (4.32) results in (4.28).  $\square$

Now, from Lemma 4.4 we get the following

**Lemma 4.5.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_1 \in V_{t_0}$ , and let there exist  $\gamma \in AC([a, t_0]; (0, +\infty))$  satisfying (4.3). Let, moreover,  $u \in AC([a, t_0]; \mathbb{R}_+)$  be such that*

$$(4.33) \quad u'(t) \geq -\ell_1(u)(t) \quad \text{for a. e. } t \in [a, t_0].$$

Then

$$(4.34) \quad \ell_1(u)(t) \leq \frac{\ell_1(\gamma)(t)}{\gamma(t)} u(t) \quad \text{for a. e. } t \in [a, t_0].$$

*Proof.* Obviously, since  $\ell_1 \in V_{t_0}$ , from (4.3) and (4.33) it follows that the assumptions of Lemma 4.4 are fulfilled for arbitrary  $\tau \in (a, t_0]$ . Therefore, according to Lemma 4.4 we have

$$\max \left\{ \frac{u(t)}{\gamma(t)} : t \in [a, \tau] \right\} = \frac{u(\tau)}{\gamma(\tau)} \quad \text{for } \tau \in [a, t_0].$$

However, the latter means that the function  $u/\gamma$  is nondecreasing. Therefore, according to Lemma 4.1 with  $\ell = \ell_1$ ,  $\alpha = \vartheta(\gamma)$ ,  $\beta = \vartheta(u/\gamma)$ , and  $\vartheta$  given by (1.6), we obtain (4.34).  $\square$

**Lemma 4.6.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $p \in L([a, t_0]; \mathbb{R}_+)$ ,  $\sigma \in \Sigma$ , (2.1) hold, and let*

$$(4.35) \quad \ell_0(1)(t) \geq p(t) \quad \text{for a. e. } t \in [a, t_0].$$

Let, moreover,  $u \in AC([a, t_0]; \mathbb{R}_+)$  satisfy

$$(4.36) \quad u'(t) \geq \ell_0(u)(t) - p(t)u(t) \quad \text{for a. e. } t \in [a, t_0],$$

and let there exist an interval  $[\tau_0, \tau_1] \subset (a, t_0]$  such that

$$(4.37) \quad u(t) > u(\tau_1) \quad \text{for } t \in [\tau_0, \tau_1].$$

Then

$$(4.38) \quad \sigma(\tau_1) < \tau_0.$$

*Proof.* Assume on the contrary that

$$(4.39) \quad \sigma(\tau_1) \geq \tau_0.$$

According to the continuity of  $u$ , in view of (4.37), there exists  $\delta \in (0, \tau_0 - a)$  such that

$$(4.40) \quad u(t) > u(\tau_1) \quad \text{for } t \in [\tau_0 - \delta, \tau_1].$$

Furthermore, the continuity of  $\sigma$ , in view of (4.39), yields the existence of  $\varepsilon > 0$  such that

$$(4.41) \quad \sigma_* \geq \tau_0 - \delta$$

where

$$(4.42) \quad \sigma_* = \min \{ \sigma(t) : t \in [\tau_1 - \varepsilon, \tau_1] \}.$$

Note that in view of (4.42) we have

$$(4.43) \quad \sigma_* \leq \sigma(\tau_1 - \varepsilon) \leq \tau_1 - \varepsilon.$$



On the other hand, from (4.36) we get

$$(4.44) \quad \left( u(t) \exp \left( - \int_t^{\tau_1} p(s) ds \right) \right)' \geq \ell_0(u)(t) \exp \left( - \int_t^{\tau_1} p(s) ds \right) \quad \text{for a. e. } t \in [a, t_0].$$

The integration of (4.44) from  $\tau_1 - \varepsilon$  to  $\tau_1$  results in

$$u(\tau_1) \geq u(\tau_1 - \varepsilon) \exp \left( - \int_{\tau_1 - \varepsilon}^{\tau_1} p(s) ds \right) + \int_{\tau_1 - \varepsilon}^{\tau_1} \ell_0(u)(s) \exp \left( - \int_s^{\tau_1} p(\xi) d\xi \right) ds,$$

whence, on account of (2.1), (4.35), and (4.40)–(4.42), we obtain

$$(4.45) \quad u(\tau_1) \geq u(\tau_1 - \varepsilon) \exp \left( - \int_{\tau_1 - \varepsilon}^{\tau_1} p(s) ds \right) + u(\tau_1) \left( 1 - \exp \left( - \int_{\tau_1 - \varepsilon}^{\tau_1} p(s) ds \right) \right).$$

However, (4.45) implies

$$u(\tau_1) \geq u(\tau_1 - \varepsilon)$$

which, on account of (4.41) and (4.43) contradicts (4.40).  $\square$

**Lemma 4.7.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $p \in L_{loc}((-\infty, t_0]; \mathbb{R}_+)$ ,  $\sigma \in \Sigma$ , and let (2.1) and (4.35) hold. Let, moreover,  $u \in AC([a, t_0]; \mathbb{R}_+)$  satisfy (4.36). Then, for every  $\tau \in (a, t_0)$ , the estimate*

$$(4.46) \quad u(t) \geq u(\tau) e^{-M_\sigma(a, t_0)} \quad \text{for } t \in [\tau, t_0]$$

*holds, where*

$$(4.47) \quad M_\sigma(a, t_0) = \max \left\{ \int_{\sigma(t)}^t p(s) ds : t \in [a, t_0] \right\}.$$

*Proof.* Assume on the contrary that (4.46) is not valid, i.e., there exist  $\tau_0 \in (a, t_0)$  and  $\tau_1 \in (\tau_0, t_0]$  such that

$$(4.48) \quad u(\tau_1) < u(\tau_0) e^{-M_\sigma(a, t_0)}.$$

Obviously, without loss of generality we can assume that (4.37) is fulfilled. Therefore, according to Lemma 4.6 we have (4.38).

On the other hand, from (4.36) we get

$$u'(t) \geq -p(t)u(t) \quad \text{for a. e. } t \in [a, t_0],$$

whence we obtain

$$u(\tau_1) \geq u(\tau_0) \exp \left( - \int_{\tau_0}^{\tau_1} p(s) ds \right).$$

However, on account of (4.38), the latter inequality yields

$$u(\tau_1) \geq u(\tau_0) \exp \left( - \int_{\sigma(\tau_1)}^{\tau_1} p(s) ds \right),$$

which, together with (4.47) contradicts (4.48).  $\square$

**Lemma 4.8.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ , and let  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  satisfy (2.6). Then (4.10) holds.*

*Proof.* Let  $\gamma_a : [a, t_0] \rightarrow (0, +\infty)$  be a restriction of  $\gamma$  to the interval  $[a, t_0]$ . From (2.6) it follows that  $\gamma$  is a nonincreasing function, and so

$$\vartheta(\gamma_a)(t) \leq \vartheta(\gamma)(t) \quad \text{for } t \in \mathbb{R}$$

where  $\vartheta$  is given by (1.6). Therefore,

$$(4.49) \quad \ell_1(\gamma_a)(t) \leq \ell_1(\gamma)(t) \quad \text{for a. e. } t \in [a, t_0].$$

Consequently, from (2.6), in view of (4.49), it follows that

$$(4.50) \quad \gamma'_a(t) \leq -\ell_1(\gamma_a)(t) \quad \text{for a. e. } t \in [a, t_0].$$

Thus, according to Proposition 4.4 the inclusion (4.10) holds.  $\square$

**Lemma 4.9.** *Let  $\ell_1 \in V_{t_0}$ ,  $\sigma \in \Sigma$ , (2.1) hold, and let there exist  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  such that (2.6)–(2.8) hold. Let, moreover,  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\kappa > 0$ , and let  $u \in AC([a, t_0]; \mathbb{R}_+)$  satisfy (4.15) and*

$$(4.51) \quad u(t_0) \leq \kappa e^{-M_\sigma}$$

where  $M_\sigma$  is given by (2.9). Then

$$(4.52) \quad u(t) \leq \kappa \quad \text{for } t \in [a, t_0].$$

*Proof.* According to Lemma 4.8 we have (4.10). Furthermore, from (2.6) it follows that  $\gamma$  is a nonincreasing function. Thus, according to Lemma 4.1 with  $\ell = \ell_1$ ,  $\alpha = \vartheta(\gamma)$ ,  $\beta = \vartheta(1/\gamma)$ ,  $\vartheta$  given by (1.6), from (2.7) it follows that (2.26) is fulfilled. Therefore, according to Lemma 4.3 we have

$$(4.53) \quad u(a) \leq u(t_0).$$

Assume that (4.52) does not hold. Then, on account of (4.51) and (4.53), there exists  $\tau \in (a, t_0)$  such that

$$(4.54) \quad u(\tau) > \kappa.$$

Let  $\gamma_a : [a, t_0] \rightarrow (0, +\infty)$  be a restriction of  $\gamma$  to the interval  $[a, t_0]$ . From the proof of Lemma 4.8 it follows that (4.49) and (4.50) hold. Therefore, according to Lemma 4.5, from (4.15) we obtain

$$(4.55) \quad u'(t) \geq \ell_0(u)(t) - \frac{\ell_1(\gamma)(t)}{\gamma(t)} u(t) \quad \text{for a. e. } t \in [a, t_0].$$

Thus, in view of (2.7), and (4.55), all the assumptions of Lemma 4.7 with  $p = \ell_1(\gamma)/\gamma$  are fulfilled. Therefore, (4.46) holds with

$$(4.56) \quad M_\sigma(a, t_0) = \max \left\{ \int_{\sigma(t)}^t \frac{\ell_1(\gamma)(s)}{\gamma(s)} ds : t \in [a, t_0] \right\}.$$

However, (2.8), (2.9), and (4.56) imply  $M_\sigma(a, t_0) \leq M_\sigma < +\infty$ , and so from (4.46) it follows that

$$(4.57) \quad u(t_0) \geq u(\tau) e^{-M_\sigma}.$$

Now (4.54) and (4.57) contradicts (4.51).  $\square$

**4.3. A priori estimates.** The following lemma can be found in [8]. We formulate it in a form suitable for us.

**Lemma 4.10.** *Let  $a, b \in \mathbb{R}$ ,  $a < b$ ,  $t_0 \in [a, b]$ , and let the problem (4.5) have on  $[a, b]$  only the trivial solution. Let, moreover, there exist  $\rho > 0$  such that every function  $u \in AC([a, b]; \mathbb{R})$  satisfying*

$$(4.58) \quad u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \lambda f(u)(t) \quad \text{for a. e. } t \in [a, b],$$

$$(4.59) \quad u(t_0) = \lambda c$$

for some  $\lambda \in (0, 1)$ , admits the estimate

$$(4.60) \quad \|u\|_{a,b} \leq \rho.$$

Then the problem (1.1), (1.2) has at least one solution on  $[a, b]$ .

**Definition 4.3.** Let  $a, b \in \mathbb{R}$ ,  $a < b$ . We say that a pair of operators  $(\ell_0, \ell_1)$  belongs to the set  $\mathcal{A}(a, b)$  if there exists  $\rho_0 > 0$  such that, for any  $q^* \in L([a, b]; \mathbb{R}_+)$  and  $c^* \in \mathbb{R}_+$ , every function  $u \in AC([a, b]; \mathbb{R})$  satisfying the inequalities

$$(4.61) \quad [u'(t) - \ell_0(u)(t) + \ell_1(u)(t)] \operatorname{sgn} u(t) \leq q^*(t) \quad \text{for a. e. } t \in [a, b],$$

$$(4.62) \quad 0 \leq u(a) \leq c^*$$

admits the estimate

$$(4.63) \quad \|u\|_{a,b} \leq \rho_0 \left( c^* + \int_a^b q^*(s) ds \right).$$

**Lemma 4.11.** *Let  $a, b \in \mathbb{R}$ ,  $a < b$ , and let  $\ell_0, \ell_1 \in V_b$ . Then  $(\ell_0, \ell_1) \in \mathcal{A}(a, b)$ .*

*Proof.* Let  $q^* \in L([a, b]; \mathbb{R}_+)$ ,  $c^* \in \mathbb{R}_+$ , and let  $u \in AC([a, b]; \mathbb{R})$  satisfy (4.61) and (4.62). Put

$$(4.64) \quad w(t) = \max \{ |u(s)| : s \in [a, t] \} \quad \text{for } t \in [a, b].$$

Then, in view of (4.61) and (4.64) we have that  $w \in AC([a, b]; \mathbb{R}_+)$ ,

$$(4.65) \quad w'(t) \geq 0 \quad \text{for a. e. } t \in [a, b],$$

$$(4.66) \quad w(t) \geq |u(t)| \quad \text{for } t \in [a, b],$$

and

$$(4.67) \quad w(a) \leq c^*.$$

Put

$$A = \{ t \in [a, b] : w(t) = |u(t)| \}.$$

Then

$$(4.68) \quad w'(t) = \begin{cases} u'(t) \operatorname{sgn} u(t) & \text{for a. e. } t \in A, \\ 0 & \text{for a. e. } t \in [a, b] \setminus A. \end{cases}$$

Furthermore, in view of (4.66), we have

$$(4.69) \quad (-1)^i \ell_i(u)(t) \operatorname{sgn} u(t) \leq \ell_i(w)(t) \quad \text{for a. e. } t \in [a, b] \quad (i = 0, 1).$$

Moreover, on account of (4.65) and the inclusions  $\ell_0, \ell_1 \in V_b$ , according to Lemma 4.1 (with  $t_0 = b$ ,  $\ell = \ell_i$ ,  $\alpha \equiv 1$ ,  $\beta = \vartheta(w)$ ,  $\vartheta$  given by (1.6)) we find

$$(4.70) \quad \ell_i(w)(t) \leq \ell_i(1)(t)w(t) \quad \text{for a. e. } t \in [a, b] \quad (i = 0, 1).$$

Thus from (4.61), on account of (4.68)–(4.70), we get

$$(4.71) \quad w'(t) \leq [\ell_0(1)(t) + \ell_1(1)(t)]w(t) + q^*(t) \quad \text{for a. e. } t \in [a, b].$$

Now, from (4.71) we obtain

$$(4.72) \quad w(b) \leq \exp \left( \int_a^b [\ell_0(1)(s) + \ell_1(1)(s)] ds \right) \times \left( w(a) + \int_a^b q^*(s) \exp \left( - \int_a^s [\ell_0(1)(\xi) + \ell_1(1)(\xi)] d\xi \right) ds \right).$$

However, from (4.72), in view of (4.65)–(4.67), it follows that (4.63) holds with

$$\rho_0 = \exp \left( \int_a^b [\ell_0(1)(s) + \ell_1(1)(s)] ds \right).$$

□

**Lemma 4.12.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_0, \ell_1, f \in V_{t_0}$ , and let (4.10) be fulfilled. Let, moreover,*

$$(4.73) \quad f(v)(t) \geq 0 \quad \text{for a. e. } t \in [a, t_0], \quad v \in C([a, t_0]; \mathbb{R}),$$

$$(4.74) \quad f(v)(t) = 0 \quad \text{for a. e. } t \in [a, t_0], \quad -v \in C([a, t_0]; \mathbb{R}_+).$$

*Then, every solution  $u$  to (1.1) on  $[a, t_0]$  satisfying*

$$(4.75) \quad u(t_0) \geq 0,$$

*admits also the inequality (4.12).*

*Proof.* First note that, according to Propositions 4.1–4.4, in view of (4.10) we have (4.4). Let  $u$  be a solution to (1.1) on  $[a, t_0]$  satisfying (4.75). It is sufficient to show that (4.11) holds, because then the assertion follows from (4.4), (4.11), and (4.73). Therefore, assume on the contrary that

$$(4.76) \quad u(a) < 0.$$

Then, in view of (4.75), there exists  $\tau \in (a, t_0]$  such that

$$(4.77) \quad u(t) < 0 \quad \text{for } t \in [a, \tau), \quad u(\tau) = 0.$$

Now, from (1.1), on account of (4.74), (4.77), and the inclusion  $f \in V_{t_0}$ , we get

$$(4.78) \quad u'(t) = \ell_0(u)(t) - \ell_1(u)(t) \quad \text{for a. e. } t \in [a, \tau].$$

However, (4.78), in view of (4.77) and the inclusion  $\ell_0 \in V_{t_0}$ , results in

$$(4.79) \quad u'(t) \leq -\ell_1(u)(t) \quad \text{for a. e. } t \in [a, \tau], \quad u(\tau) = 0.$$

According to Remark 4.1, the inclusion (4.10) yields (4.23), which together with (4.79) implies

$$(4.80) \quad u(t) \geq 0 \quad \text{for } t \in [a, \tau].$$

However, the inequality (4.80) contradicts (4.76). □

**Lemma 4.13.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_0, \ell_1, f \in V_{t_0}$ , and let  $\ell_0 - \ell_1 \in \mathcal{P}_{t_0}^+$ . Let, moreover, (4.73) and (4.74) be fulfilled. Then, every solution  $u$  to (1.1) on  $[a, t_0]$  satisfying (4.75) admits also the inequalities (4.12) and*

$$(4.81) \quad u'(t) \geq 0 \quad \text{for a. e. } t \in [a, t_0].$$

*Proof.* First note that, according to Propositions 4.1 and 4.6, we have (4.6). Let  $u$  be a solution to (1.1) on  $[a, t_0]$  satisfying (4.75). It is sufficient to show that (4.11) holds, because then the assertion follows from (4.6), (4.11), and (4.73). Therefore, assume on the contrary that (4.76) holds. Then, in view of (4.75), there exists  $\tau \in (a, t_0]$  such that (4.77) is satisfied. Now, from (1.1), on account of (4.74), (4.77), and the inclusion  $f \in V_{t_0}$ , we get (4.78). However, the inclusion  $\ell_0 - \ell_1 \in \mathcal{P}_{t_0}^+$  yields  $\ell_0 - \ell_1 \in \mathcal{P}_\tau^+$ . Moreover, the inclusion  $\ell_0 \in V_{t_0}$  implies  $\ell_0 \in V_\tau$ , and so, according to Proposition 4.1, we have  $\ell_0 \in \mathcal{S}_{a\tau}(a)$ . Consequently, according to Proposition 4.6 (with  $t_0 = \tau$ ), we have  $\ell_0 - \ell_1 \in \mathcal{S}'_{a\tau}(a)$ , which together with (4.76) and (4.78), implies  $u(\tau) \leq u(a) < 0$ . However, the latter contradicts (4.77).  $\square$

**Lemma 4.14.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_0, \ell_1, f \in V_{t_0}$ , (4.10) hold, and let there exist  $q \in L([a, t_0]; \mathbb{R}_+)$  such that*

$$(4.82) \quad f(v)(t) \operatorname{sgn} v(t) \leq q(t) \quad \text{for a. e. } t \in [a, t_0], \quad v \in C([a, t_0]; \mathbb{R})$$

*is fulfilled. Let, moreover, (2.26), (4.73), and (4.74) hold. Then, for every  $c \in \mathbb{R}_+$ , the problem (1.1), (1.2) has at least one solution  $u$  on  $[a, t_0]$  satisfying (4.12).*

*Proof.* Let  $c \in \mathbb{R}_+$  be arbitrary but fixed. According to Lemmas 4.2, 4.10, and 4.12, it is sufficient to show that there exists  $\rho > 0$  such that every function  $u \in AC([a, t_0]; \mathbb{R})$  satisfying (4.59) and

$$(4.83) \quad u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \lambda f(u)(t) \quad \text{for a. e. } t \in [a, t_0],$$

for some  $\lambda \in (0, 1)$ , admits the estimate

$$(4.84) \quad \|u\|_{a, t_0} \leq \rho.$$

Let, therefore,  $\lambda \in (0, 1)$  and let  $u \in AC([a, t_0]; \mathbb{R})$  satisfy (4.59) and (4.83). Then, in view of (4.73) we have (4.15) and, according to Lemma 4.12 we have (4.12). Thus, on account of (2.26), (4.10), (4.12), and (4.15), all the assumptions of Lemma 4.3 are fulfilled. Therefore, (4.53) holds. Finally, according to Lemma 4.11, in view of (4.12), (4.53), (4.59), (4.82), and (4.83), there exists  $\rho_0 > 0$  such that

$$\|u\|_{a, t_0} \leq \rho_0 \left( c + \int_a^{t_0} q(s) ds \right)$$

holds. Consequently, (4.84) is fulfilled with  $\rho = \rho_0 \left( c + \int_a^{t_0} q(s) ds \right)$ .  $\square$

**Lemma 4.15.** *Let  $a \in \mathbb{R}$ ,  $a < t_0$ ,  $\ell_0, \ell_1, f \in V_{t_0}$ ,  $\ell_0 - \ell_1 \in \mathcal{P}_{t_0}^+$ . Let, moreover, (4.73) and (4.74) be satisfied. Then, for every  $c \in \mathbb{R}_+$ , the problem (1.1), (1.2) has at least one solution  $u$  on  $[a, t_0]$  satisfying (4.12) and (4.81).*

*Proof.* Let  $c \in \mathbb{R}_+$  be arbitrary but fixed. According to Propositions 4.1 and 4.5, and Lemmas 4.10 and 4.13, it is sufficient to show that there exists  $\rho > 0$  such that every function  $u \in AC([a, t_0]; \mathbb{R})$  satisfying (4.59) and (4.83) for some  $\lambda \in (0, 1)$ , admits the estimate (4.84). Let, therefore,  $\lambda \in (0, 1)$  and let  $u \in AC([a, t_0]; \mathbb{R})$  satisfy (4.59) and (4.83). Then, according to Lemma 4.13 we have (4.12) and (4.81). Therefore, on account of (4.59), the estimate (4.84) is fulfilled with  $\rho = c$ .  $\square$

**Lemma 4.16.** *Let  $c \in \mathbb{R}_+$ ,  $b \in \mathbb{R}$ ,  $b > t_0$ , and let  $(\ell_0, \ell_1) \in \mathcal{A}(t_0, b)$ . Let, moreover,*

$$(4.85) \quad f(v)(t) \operatorname{sgn} v(t) \leq q(t, \|v\|_{t_0, b}) \quad \text{for a. e. } t \in [t_0, b], \quad v \in C([t_0, b]; \mathbb{R}), \quad 0 \leq v(t_0) \leq c,$$

*where  $q \in K([t_0, b] \times \mathbb{R}_+; \mathbb{R}_+)$  satisfies (2.5). Then the problem (1.1), (1.2) has at least one solution on  $[t_0, b]$ .*

*Proof.* First note that due to the inclusion  $(\ell_0, \ell_1) \in \mathcal{A}(t_0, b)$ , the problem (4.5) has on  $[t_0, b]$  only the trivial solution.

Let  $\rho_0$  be the number appearing in Definition 4.3. According to (2.5) there exists  $\rho > 2c\rho_0$  such that

$$(4.86) \quad \frac{1}{x} \int_{t_0}^b q(s, x) ds < \frac{1}{2\rho_0} \quad \text{for } x > \rho.$$

Now assume that a function  $u \in AC([t_0, b]; \mathbb{R})$  satisfies (4.59) and

$$(4.87) \quad u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \lambda f(u)(t) \quad \text{for a. e. } t \in [t_0, b]$$

for some  $\lambda \in (0, 1)$ . Then, according to (4.85) we obtain that

$$(4.88) \quad [u'(t) - \ell_0(u)(t) + \ell_1(u)(t)] \operatorname{sgn} u(t) \leq q(t, \|u\|_{t_0, b}) \quad \text{for a. e. } t \in [t_0, b],$$

$$(4.89) \quad 0 \leq u(t_0) \leq c.$$

Hence, by the inclusion  $(\ell_0, \ell_1) \in \mathcal{A}(t_0, b)$  and (4.86), we get the estimate

$$(4.90) \quad \|u\|_{t_0, b} \leq \rho.$$

Since  $\rho$  depends neither on  $u$  nor on  $\lambda$ , it follows from Lemma 4.10 that the problem (1.1), (1.2) has at least one solution on  $[t_0, b]$ .  $\square$

**4.4. Existence of solutions defined on half-lines and on  $\mathbb{R}$ .** To formulate the following lemma we need to introduce some notation. Let  $\kappa > 0$ ,  $I \subseteq \mathbb{R}$  be a closed interval. Then, for every continuous function  $v : I \rightarrow \mathbb{R}$ , we put

$$(4.91) \quad \bar{f}(v)(t) \stackrel{\text{def}}{=} f(\psi(\vartheta(v)))(t) \quad \text{for a. e. } t \in \mathbb{R},$$

where  $\vartheta$  is given by (1.6) and

$$(4.92) \quad \psi(v)(t) = \begin{cases} \kappa & \text{if } v(t) > \kappa, \\ v(t) & \text{if } 0 \leq v(t) \leq \kappa, \\ 0 & \text{if } v(t) < 0 \end{cases} \quad \text{for } t \in \mathbb{R}.$$

Note that  $\bar{f} \in V_{t_0}$  provided  $f \in V_{t_0}$ . Let  $(a_n)_{n=1}^{+\infty}$  be a sequence of real numbers such that

$$(4.93) \quad a_n < t_0, \quad \lim_{n \rightarrow +\infty} a_n = -\infty,$$

and consider the auxiliary equation

$$(4.94) \quad u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \bar{f}(u)(t).$$

**Lemma 4.17.** *Let  $\ell_0, \ell_1 \in V_{t_0}$ . Let, moreover, there exist  $\kappa > 0$  and  $c \in [0, \kappa]$  such that, for every  $n \in \mathbb{N}$ , the problem (4.94), (1.2) has a solution  $u_n$  on the interval  $[a_n, t_0]$  satisfying*

$$(4.95) \quad 0 \leq u_n(t) \leq \kappa \quad \text{for } t \in [a_n, t_0].$$

*Then (1.1), (1.2) has at least one solution  $u$  on  $(-\infty, t_0]$  satisfying (2.10). If, in addition,*

$$(4.96) \quad u'_n(t) \geq 0 \quad \text{for a. e. } t \in [a_n, t_0], \quad n \in \mathbb{N}$$

*then (1.1), (1.2) has at least one solution  $u$  on  $(-\infty, t_0]$  satisfying (2.10) and (2.11).*

*Proof.* Let  $a \in \mathbb{R}$  be arbitrary but fixed such that  $a < t_0$ . Then, in view of (4.93) there exists  $n_0 \in \mathbb{N}$  such that  $[a, t_0] \subseteq [a_n, t_0]$  for  $n \geq n_0$ . Further, in view of (4.91) and (4.95), there exists  $q \in L([a, t_0]; \mathbb{R}_+)$  such that

$$(4.97) \quad |\bar{f}(u_n)(t)| \leq q(t) \quad \text{for a. e. } t \in [a, t_0], \quad n \geq n_0.$$

Thus (4.94), on account of (4.95), and (4.97), results in

$$(4.98) \quad |u'_n(t)| \leq [\ell_0(1)(t) + \ell_1(1)(t)]\kappa + q(t) \quad \text{for a. e. } t \in [a, t_0], \quad n \geq n_0.$$

Therefore, on account of (4.95) and (4.98), the sequence of solutions  $(u_n)_{n=n_0}^{+\infty}$  is uniformly bounded and equicontinuous on  $[a, t_0]$ . Since the interval  $[a, t_0]$  was chosen arbitrarily, according to Arzelà-Ascoli theorem, without loss of generality we can assume that there exists  $u \in C_{loc}(\mathbb{R}; \mathbb{R})$  such that

$$(4.99) \quad \lim_{n \rightarrow +\infty} \vartheta(u_n)(t) = u(t) \quad \text{uniformly on every compact interval.}$$

On the other hand, from (1.2) and (4.94), in view of (4.91)–(4.93) and (4.95) we get

$$(4.100) \quad u_n(t) = c - \int_t^{t_0} [\ell_0(u_n)(s) - \ell_1(u_n)(s) + f(u_n)(s)] ds \quad \text{for } t \in [a, t_0], \quad n \geq n_0.$$

Thus (4.99) and (4.100) yields

$$u(t) = c - \int_t^{t_0} [\ell_0(u)(s) - \ell_1(u)(s) + f(u)(s)] ds \quad \text{for } t \in [a, t_0].$$

Therefore, because the interval  $[a, t_0]$  was chosen arbitrarily, we have that  $u \in AC_{loc}(\mathbb{R}; \mathbb{R})$  (note that, according to (4.99),  $u(t) = c$  for  $t \geq t_0$ ) and the restriction of  $u$  to the interval  $(-\infty, t_0]$  is a solution to the problem (1.1), (1.2) on  $(-\infty, t_0]$ . Obviously, according to (4.95),  $u$  satisfies (2.10) being a limit of  $\vartheta(u_n)$ .

Moreover, if (4.96) holds then, for every  $n \in \mathbb{N}$ ,

$$\vartheta(u_n)(\tau_1) \leq \vartheta(u_n)(\tau_2) \quad \text{whenever } \tau_1 \leq \tau_2,$$

and so, in view of (4.99), we have

$$u(\tau_1) \leq u(\tau_2) \quad \text{whenever } \tau_1 \leq \tau_2.$$

Therefore, (2.11) holds. □

**Lemma 4.18.** *Let  $\ell_0, \ell_1, f \in V$ ,  $c \in \mathbb{R}_+$ , and let*

$$(4.101) \quad f(v)(t) \operatorname{sgn} v(t) \leq q(t, \|\vartheta(v)\|) \quad \text{for a. e. } t \geq t_0,$$

$$v \in C_0([t_0, +\infty); \mathbb{R}), \quad 0 \leq v(t_0) \leq c,$$

where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  satisfies (2.5) for every  $b > t_0$ . Then the problem (1.1), (1.2) has at least one solution on  $[t_0, +\infty)$ .

*Proof.* Note that, in view of (4.101), we have that (4.85) holds for every  $b \in \mathbb{R}$ ,  $b > t_0$ . Therefore, according to Lemmas 4.11 and 4.16, the problem (1.1), (1.2) has at least one solution on  $[t_0, b]$  for every  $b > t_0$ .

Let  $(b_n)_{n=1}^{+\infty}$  be a sequence of real numbers such that

$$(4.102) \quad b_n > t_0, \quad \lim_{n \rightarrow +\infty} b_n = +\infty,$$

and let, for every  $n \in \mathbb{N}$ ,  $u_n$  be a solution to (1.1), (1.2) on  $[t_0, b_n]$ . Let, moreover,  $b \in \mathbb{R}$  be arbitrary but fixed such that  $b > t_0$ . Then, in view of (4.102), there exists  $n_0 \in \mathbb{N}$  such that

$[t_0, b] \subseteq [t_0, b_n]$  for  $n \geq n_0$ . Further, let, for every  $n \geq n_0$ ,  $\bar{u}_n$  be a restriction of  $u_n$  to the interval  $[t_0, b]$ . Then, in view of the inclusion  $\ell_0, \ell_1, f \in V$  we have

$$(4.103) \quad \bar{u}'_n(t) = \ell_0(\bar{u}_n)(t) - \ell_1(\bar{u}_n)(t) + f(\bar{u}_n)(t) \quad \text{for a. e. } t \in [t_0, b], \quad n \geq n_0,$$

$$(4.104) \quad \bar{u}_n(t_0) = c \quad \text{for } n \geq n_0.$$

According to Lemma 4.11 we have  $(\ell_0, \ell_1) \in \mathcal{A}(t_0, b)$ . Let  $\rho_0$  be the number appearing in Definition 4.3. According to (2.5) there exists  $\rho > 2c\rho_0$  such that (4.86) holds. Thus, according to (4.85), (4.103), and (4.104), for every  $n \geq n_0$  we obtain

$$\begin{aligned} [\bar{u}'_n(t) - \ell_0(\bar{u}_n)(t) + \ell_1(\bar{u}_n)(t)] \operatorname{sgn} \bar{u}_n(t) &\leq q(t, \|\bar{u}_n\|_{t_0, b}) \quad \text{for a. e. } t \in [t_0, b], \\ 0 &\leq \bar{u}_n(t_0) \leq c. \end{aligned}$$

Hence, according to  $(\ell_0, \ell_1) \in \mathcal{A}(t_0, b)$  and (4.86), we get the estimate

$$(4.105) \quad \|\bar{u}_n\|_{t_0, b} \leq \rho \quad \text{for } n \geq n_0.$$

Moreover, using (4.105) in (4.103) we get

$$(4.106) \quad |\bar{u}'_n(t)| \leq [\ell_0(1)(t) + \ell_1(1)(t)]\rho + f^*(t) \quad \text{for a. e. } t \in [t_0, b], \quad n \geq n_0$$

where  $f^* \in L([t_0, b]; \mathbb{R}_+)$  is such that

$$|f(v)(t)| \leq f^*(t) \quad \text{for a. e. } t \in [t_0, b], \quad v \in C([t_0, b]; \mathbb{R}), \quad \|v\|_{t_0, b} \leq \rho.$$

Consequently, from (4.105) and (4.106) it follows that the sequence of solutions  $(u_n)_{n=n_0}^{+\infty}$  is uniformly bounded and equicontinuous on  $[t_0, b]$ . Since the interval  $[t_0, b]$  was chosen arbitrarily, according to Arzelà-Ascoli theorem, without loss of generality we can assume that there exists  $u \in C_{loc}(\mathbb{R}; \mathbb{R})$  such that (4.99) holds.

On the other hand, from (1.1) and (1.2), we have

$$(4.107) \quad u_n(t) = c + \int_{t_0}^t [\ell_0(u_n)(s) - \ell_1(u_n)(s) + f(u_n)(s)] ds \quad \text{for } t \in [t_0, b], \quad n \geq n_0.$$

Thus, (4.99) and (4.107) yield

$$u(t) = c + \int_{t_0}^t [\ell_0(u)(s) - \ell_1(u)(s) + f(u)(s)] ds \quad \text{for } t \in [t_0, b].$$

Therefore, because the interval  $[t_0, b]$  was chosen arbitrarily, we have that  $u \in AC_{loc}(\mathbb{R}; \mathbb{R})$  (note that, according to (4.99),  $u(t) = c$  for  $t \leq t_0$ ) and the restriction of  $u$  to the interval  $[t_0, +\infty)$  is a solution to the problem (1.1), (1.2) on  $[t_0, +\infty)$ .  $\square$

**Lemma 4.19.** *Let  $\ell_0, \ell_1, f \in V$ ,  $\kappa > 0$ ,  $c \in [0, \kappa]$ , and let*

$$(4.108) \quad \begin{aligned} f(v)(t) \operatorname{sgn} v(t) &\leq q(t, \|v\|) \quad \text{for a. e. } t \geq t_0, \quad v \in C_0(\mathbb{R}; \mathbb{R}), \\ 0 &\leq v(t_0) \leq c, \quad -c \leq v(t) \leq \kappa \quad \text{for } t \leq t_0, \end{aligned}$$

where  $q \in K_{loc}([t_0, +\infty) \times \mathbb{R}_+; \mathbb{R}_+)$  is nondecreasing in the second argument and satisfies (2.5) for every  $b > t_0$ . Let, moreover, there exist a solution  $u_0$  to (1.1), (1.2) on  $(-\infty, t_0]$  satisfying

$$(4.109) \quad 0 \leq u_0(t) \leq \kappa \quad \text{for } t \leq t_0.$$

Then the problem (1.1), (1.2) has at least one global solution  $u$  such that

$$u(t) = u_0(t) \quad \text{for } t \leq t_0.$$



*Proof.* Consider the auxiliary equation

$$(4.110) \quad u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \widehat{f}(u)(t)$$

where

$$(4.111) \quad \widehat{f}(v)(t) \stackrel{\text{def}}{=} f(\vartheta(u_0) - c + v)(t) + \ell_0(\vartheta(u_0) - c)(t) - \ell_1(\vartheta(u_0) - c)(t) \\ \text{for a. e. } t \in \mathbb{R}, \quad v \in C_{loc}(\mathbb{R}; \mathbb{R})$$

where  $\vartheta$  is given by (1.6). Obviously,  $\widehat{f} \in V$ .

Now let  $v \in C_0([t_0, +\infty); \mathbb{R})$  be arbitrary but fixed such that

$$(4.112) \quad 0 \leq v(t_0) \leq c$$

and put

$$(4.113) \quad w(t) = \vartheta(u_0)(t) - c + \vartheta(v)(t) \quad \text{for } t \in \mathbb{R},$$

$$(4.114) \quad \widehat{q}(t, x) = q(t, x + \kappa) + \kappa [\ell_0(1)(t) + \ell_1(1)(t)] \quad \text{for a. e. } t \geq t_0, \quad x \in \mathbb{R}_+.$$

Then, in view of (4.109), (4.112), and (4.113) we have  $w \in C_0(\mathbb{R}; \mathbb{R})$ ,

$$(4.115) \quad 0 \leq w(t_0) \leq c,$$

$$(4.116) \quad -c \leq w(t) \leq \kappa \quad \text{for } t \leq t_0,$$

$$(4.117) \quad \|\vartheta(u_0) - c\| \leq \kappa.$$

Consequently, in view of (4.108) and (4.114)–(4.117) we have

$$(4.118) \quad f(w)(t) \operatorname{sgn} w(t) \leq q(t, \|w\|) \leq q(t, \|\vartheta(v)\| + \kappa) \quad \text{for a. e. } t \geq t_0.$$

On the other hand,

$$(4.119) \quad \widehat{f}(v)(t) \operatorname{sgn} v(t) = [f(w)(t) + \ell_0(\vartheta(u_0) - c)(t) - \ell_1(\vartheta(u_0) - c)(t)] \operatorname{sgn} w(t) \quad \text{for a. e. } t \geq t_0.$$

Therefore, (4.118) and (4.119), in view of (4.112), (4.114), and (4.117), result in

$$\widehat{f}(v)(t) \operatorname{sgn} v(t) \leq \widehat{q}(t, \|\vartheta(v)\|) \quad \text{for a. e. } t \geq t_0, \quad v \in C_0([t_0, +\infty); \mathbb{R}), \quad 0 \leq v(t_0) \leq c.$$

Moreover, on account of (4.114),

$$\frac{1}{x} \int_{t_0}^b \widehat{q}(t, x) dt = \left(1 + \frac{\kappa}{x}\right) \frac{1}{x + \kappa} \int_{t_0}^b q(t, x + \kappa) dt + \frac{\kappa}{x} \int_{t_0}^b [\ell_0(1)(t) + \ell_1(1)(t)] dt \quad \text{for } b > t_0,$$

and thus, in view of (2.5), we have

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \int_{t_0}^b \widehat{q}(t, x) dt = 0$$

for every  $b > t_0$ . Consequently, all the assumptions of Lemma 4.18 are fulfilled with  $f = \widehat{f}$  and  $q = \widehat{q}$ . Therefore, the problem (4.110), (1.2) has at least one solution  $u_1$  on  $[t_0, +\infty)$ .

Now put

$$u(t) = \begin{cases} u_0(t) & \text{for } t \leq t_0, \\ u_1(t) & \text{for } t > t_0. \end{cases}$$

Then, obviously,  $u \in AC_{loc}(\mathbb{R}; \mathbb{R})$  and in view of (4.110) and (4.111), the function  $u$  is a global solution to (1.1), (1.2).  $\square$

#### 4.5. Properties of a solution in the neighbourhood of $-\infty$ .

**Lemma 4.20.** *Let  $\ell_0, f \in V_{t_0}$ , and let there exist  $g \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$  and a continuous nondecreasing function  $h_0 : (0, +\infty) \rightarrow (0, +\infty)$  satisfying (2.12) and (2.13). Let, moreover,  $u$  be a non-negative solution to (1.1) on  $(-\infty, t_0]$  such that*

$$(4.120) \quad u(t) = 0 \quad \text{for } t \leq \tau,$$

for some  $\tau \in (-\infty, t_0)$ . Then  $u \equiv 0$ .

*Proof.* Suppose on the contrary that  $u$  assumes positive values. Put

$$(4.121) \quad w(t) = \sup \{u(s) : s \leq t\} \quad \text{for } t \leq t_0.$$

Then

$$(4.122) \quad w \in AC_{loc}((-\infty, t_0]; \mathbb{R}_+),$$

$$(4.123) \quad w'(t) \geq 0 \quad \text{for a. e. } t \leq t_0,$$

$$(4.124) \quad w(t) \geq u(t) \quad \text{for } t \leq t_0,$$

and there exists  $\tau_0 \in [\tau, t_0)$  such that

$$(4.125) \quad w(t) = 0 \quad \text{for } t \leq \tau_0, \quad w(t) > 0 \quad \text{for } t \in (\tau_0, t_0].$$

Let

$$A = \{t \in [\tau_0, t_0] : w(t) = u(t)\}.$$

Then

$$(4.126) \quad w'(t) = \begin{cases} u'(t) & \text{for a. e. } t \in A, \\ 0 & \text{for a. e. } t \in [\tau_0, t_0] \setminus A. \end{cases}$$

Furthermore, in view of (4.124), we have

$$(4.127) \quad \ell_0(u)(t) \leq \ell_0(w)(t) \quad \text{for a. e. } t \in [\tau_0, t_0],$$

and, on account of (4.123), the non-negativity of  $u$ , and the inclusion  $\ell_0 \in V_{t_0}$ , according to Lemma 4.1 (with  $\ell = \ell_0$ ,  $\alpha \equiv 1$ ,  $\beta = \vartheta(w)$ ,  $\vartheta$  given by (1.6)), we find

$$(4.128) \quad \ell_0(w)(t) \leq \ell_0(1)(t)w(t) \quad \text{for a. e. } t \in [\tau_0, t_0],$$

$$(4.129) \quad \ell_1(u)(t) \geq 0 \quad \text{for a. e. } t \in [\tau_0, t_0].$$

Moreover, from (2.12), in view of (4.121), (4.125), the non-negativity of  $u$ , and the inclusion  $f \in V_{t_0}$ , for every fixed  $t \in (\tau_0, t_0]$  we have

$$(4.130) \quad f(u)(s) \leq g(s)h_0(\|u\|_{\tau_0, t}) = g(s)h_0(w(t)) \quad \text{for a. e. } s \in [\tau_0, t].$$

Consequently, analogously to the proof of Lemma 4.1, one can show that (4.130) implies

$$(4.131) \quad f(u)(t) \leq g(t)h_0(w(t)) \quad \text{for a. e. } t \in (\tau_0, t_0].$$

Thus, since  $w$ ,  $g$ , and  $h_0$  are non-negative functions, from (1.1), (4.126)–(4.129) and (4.131) we get

$$(4.132) \quad w'(t) \leq \ell_0(1)(t)w(t) + g(t)h_0(w(t)) \quad \text{for a. e. } t \in (\tau_0, t_0].$$

However, (4.132) results in

$$(4.133) \quad z'(t) \leq g(t) \exp \left( \int_t^{t_0} \ell_0(1)(s) ds \right) \times \\ h_0 \left( z(t) \exp \left( - \int_t^{t_0} \ell_0(1)(s) ds \right) \right) \quad \text{for a. e. } t \in (\tau_0, t_0]$$

where

$$(4.134) \quad z(t) = w(t) \exp \left( \int_t^{t_0} \ell_0(1)(s) ds \right) \quad \text{for } t \in (\tau_0, t_0].$$

Since  $h_0$  is a nondecreasing function, on account of (4.125) and (4.134), from (4.133) it follows that

$$(4.135) \quad \frac{z'(t)}{h_0(z(t))} \leq g(t) \exp \left( \int_t^{t_0} \ell_0(1)(s) ds \right) \quad \text{for a. e. } t \in (\tau_0, t_0].$$

Now, the integration of (4.135) from  $t$  to  $t_0$  yields

$$\int_{z(t)}^{z(t_0)} \frac{ds}{h_0(s)} \leq \int_t^{t_0} g(s) \exp \left( \int_s^{t_0} \ell_0(1)(\xi) d\xi \right) ds \quad \text{for } t \in (\tau_0, t_0]$$

whence we obtain

$$(4.136) \quad \lim_{t \rightarrow \tau_0+} \int_{z(t)}^{z(t_0)} \frac{ds}{h_0(s)} \leq \int_{\tau_0}^{t_0} g(s) \exp \left( \int_s^{t_0} \ell_0(1)(\xi) d\xi \right) ds < +\infty.$$

However, (4.136) together with (4.122), (4.125), and (4.134) contradicts (2.13).  $\square$

**Lemma 4.21.** *Let  $\tau \in \mathbb{R}$  and let there exist  $\gamma \in AC_{loc}((-\infty, \tau]; (0, +\infty))$  satisfying*

$$(4.137) \quad \gamma'(t) \leq -\ell_1(\gamma)(t) \quad \text{for a. e. } t \leq \tau.$$

*Let, moreover,  $u \in AC_{loc}((-\infty, \tau]; \mathbb{R})$  satisfy*

$$(4.138) \quad 0 \leq \sup \{u(t) : t \leq \tau\} < +\infty,$$

$$(4.139) \quad u'(t) \geq -\ell_1(u)(t) \quad \text{for a. e. } t \leq \tau.$$

*Then*

$$(4.140) \quad \sup \left\{ \frac{u(t)}{\gamma(t)} : t \leq \tau \right\} = \frac{u(\tau)}{\gamma(\tau)}.$$

*Proof.* First suppose that  $u(t) \leq 0$  for  $t \leq \tau$ . Then, in view of (4.139),  $u$  is a nondecreasing function, which together with (4.138) implies  $u(\tau) = 0$ . Therefore, in that case (4.140) holds.

Let, therefore,  $u$  assumes positive values. Put

$$(4.141) \quad \lambda = \sup \left\{ \frac{u(t)}{\gamma(t)} : t \leq \tau \right\}.$$

Then, according to (4.137)–(4.139) we have  $0 < \lambda < +\infty$ ,

$$(4.142) \quad \lambda \gamma(t) - u(t) \geq 0 \quad \text{for } t \leq \tau,$$

$$(4.143) \quad \lambda \gamma'(t) - u'(t) \leq -\ell_1(\lambda \gamma - u)(t) \quad \text{for a. e. } t \leq \tau.$$

According to (4.137),  $\gamma$  is a nonincreasing function. Therefore, there exists finite or infinite limit  $\gamma(-\infty)$ . If

$$(4.144) \quad \gamma(-\infty) = +\infty$$

then, in view of (4.138) and (4.141), there exists  $\tau_0 \in (-\infty, \tau]$  such that (4.32) holds. If

$$(4.145) \quad \gamma(-\infty) < +\infty$$

then, on account of (4.141), for every  $n \in \mathbb{N}$  there exists  $\tau_n \in (-\infty, \tau]$  such that

$$\lambda - \frac{1}{n\gamma(-\infty)} \leq \frac{u(\tau_n)}{\gamma(\tau_n)}$$

whence, because  $\gamma$  is nonincreasing, we get

$$\lambda\gamma(\tau_n) - u(\tau_n) \leq \frac{\gamma(\tau_n)}{n\gamma(-\infty)} \leq \frac{1}{n}.$$

Thus, in both cases (4.144) and (4.145), for every  $n \in \mathbb{N}$  there exists  $\tau_n \in (-\infty, \tau]$  such that

$$(4.146) \quad \lambda\gamma(\tau_n) - u(\tau_n) \leq \frac{1}{n}.$$

However, from (4.142) and (4.143) it follows that  $\lambda\gamma - u$  is a nonincreasing function, which together with (4.146) implies

$$(4.147) \quad \lambda\gamma(\tau) - u(\tau) \leq \frac{1}{n} \quad \text{for } n \in \mathbb{N}.$$

Now (4.141), (4.142), and (4.147) results in (4.140).  $\square$

Now, from Lemma 4.21 we get the following

**Lemma 4.22.** *Let  $\ell_1 \in V_{t_0}$ , and let there exist  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  such that (2.6) holds. Further, let  $u \in AC_{loc}((-\infty, t_0]; \mathbb{R}_+)$  satisfy*

$$(4.148) \quad u'(t) \geq -\ell_1(u)(t) \quad \text{for a. e. } t \leq t_0,$$

$$(4.149) \quad \sup \{u(t) : t \leq t_0\} < +\infty.$$

*Then*

$$(4.150) \quad \ell_1(u)(t) \leq \frac{\ell_1(\gamma)(t)}{\gamma(t)} u(t) \quad \text{for a. e. } t \leq t_0.$$

*Proof.* Obviously, since  $\ell_1 \in V_{t_0}$ , according to Lemma 4.21 we have

$$\sup \left\{ \frac{u(t)}{\gamma(t)} : t \leq \tau \right\} = \frac{u(\tau)}{\gamma(\tau)} \quad \text{for } \tau \leq t_0.$$

However, the latter means that the function  $u/\gamma$  is nondecreasing. Therefore, according to Lemma 4.1 with  $\ell = \ell_1$ ,  $\alpha = \vartheta(\gamma)$ ,  $\beta = \vartheta(u/\gamma)$ , and  $\vartheta$  given by (1.6), we obtain (4.150).  $\square$

The other assertion which can be deduced from Lemma 4.21 is the following

**Lemma 4.23.** *Let  $\tau \in \mathbb{R}$  and let there exist  $\gamma \in AC_{loc}((-\infty, \tau]; (0, +\infty))$  such that (4.137) is fulfilled. Further, let  $u \in AC_{loc}((-\infty, \tau]; \mathbb{R})$  satisfy (4.139),*

$$\sup \{u(t) : t \leq \tau\} < +\infty,$$

*and*

$$(4.151) \quad u(\tau) \leq 0.$$

*Then*

$$(4.152) \quad u(t) \leq 0 \quad \text{for } t \leq \tau.$$

*Proof.* Assume on the contrary that there exists  $\tau_0 \in (-\infty, \tau)$  such that

$$(4.153) \quad u(\tau_0) > 0.$$

Then, according to Lemma 4.21 we have (4.140). However, (4.140) together with (4.151) contradicts (4.153).  $\square$

Analogously to Lemma 4.6 one can prove the following

**Lemma 4.24.** *Let  $p \in L_{loc}((-\infty, t_0]; \mathbb{R}_+)$ ,  $\sigma \in \Sigma$ , (2.1) hold, and let*

$$(4.154) \quad \ell_0(1)(t) \geq p(t) \quad \text{for a. e. } t \leq t_0.$$

*Let, moreover,  $u \in AC_{loc}((-\infty, t_0]; \mathbb{R}_+)$  satisfy*

$$(4.155) \quad u'(t) \geq \ell_0(u)(t) - p(t)u(t) \quad \text{for a. e. } t \leq t_0,$$

*and let there exist an interval  $[\tau_0, \tau_1] \subset (-\infty, t_0]$  such that (4.37) is fulfilled. Then (4.38) holds.*

**Lemma 4.25.** *Let  $p \in L_{loc}((-\infty, t_0]; \mathbb{R}_+)$ ,  $\sigma \in \Sigma$ , (2.1) and (4.154) hold, and let*

$$(4.156) \quad \sup \left\{ \int_{\sigma(t)}^t p(s) ds : t \leq t_0 \right\} < +\infty.$$

*Let, moreover,  $u \in AC_{loc}((-\infty, t_0]; \mathbb{R}_+)$  satisfy (4.149) and (4.155). Then there exists a (finite) limit  $u(-\infty)$ .*

*Proof.* Assume on the contrary that

$$(4.157) \quad u^* - u_* > 0$$

where

$$(4.158) \quad u_* = \liminf_{t \rightarrow -\infty} u(t), \quad u^* = \limsup_{t \rightarrow -\infty} u(t).$$

In view of (4.156) and (4.157) there exists  $\delta > 0$  such that

$$(4.159) \quad 2\delta < (u^* - u_*)e^{-M_\sigma}$$

where

$$(4.160) \quad M_\sigma = \sup \left\{ \int_{\sigma(t)}^t p(s) ds : t \leq t_0 \right\}.$$

Then, in view of (4.158), there exists  $t_\delta \leq t_0$  such that

$$(4.161) \quad u(t) \geq u_* - \delta \quad \text{for } t \leq t_\delta.$$

Further, according to (4.158) there exist  $\tau_0 < \tau_1 \leq t_\delta$  such that

$$(4.162) \quad u(\tau_0) \geq u^* - \delta, \quad u(\tau_1) \leq u_* + \delta,$$

and, obviously, without loss of generality we can assume that (4.37) holds. Thus, according to Lemma 4.24 we have (4.38).

On the other hand, from (4.155) we get

$$u(\tau_1) \geq u(\tau_0) \exp \left( - \int_{\tau_0}^{\tau_1} p(s) ds \right) + \int_{\tau_0}^{\tau_1} \ell_0(u)(s) \exp \left( - \int_s^{\tau_1} p(\xi) d\xi \right) ds,$$

whence, on account of (2.1), (4.154), (4.161), and (4.162) we find

$$(4.163) \quad u_* + \delta \geq (u^* - \delta) \exp \left( - \int_{\tau_0}^{\tau_1} p(s) ds \right) + (u_* - \delta) \left( 1 - \exp \left( - \int_{\tau_0}^{\tau_1} p(s) ds \right) \right).$$

Now (4.163) results in

$$2\delta \geq (u^* - u_*) \exp \left( - \int_{\tau_0}^{\tau_1} p(s) ds \right)$$

which, together with (4.38), (4.157), and (4.160) contradicts (4.159).  $\square$

**Lemma 4.26.** *Let  $\ell_1 \in V_{t_0}$ , (2.26) hold, and let there exist  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  satisfying (2.6). Further, let  $u \in AC_{loc}((-\infty, t_0]; \mathbb{R}_+)$  satisfy*

$$(4.164) \quad u'(t) \geq \ell_0(u)(t) - \ell_1(u)(t) \quad \text{for a. e. } t \leq t_0,$$

*and assume that there exists a finite limit  $u(-\infty)$ . Then*

$$(4.165) \quad u(t) \geq u(-\infty) \quad \text{for } t \leq t_0,$$

*and, in addition, if there exists  $\tau \in (-\infty, t_0]$  such that  $u(\tau) = u(-\infty)$ , then*

$$(4.166) \quad u(t) = u(-\infty) \quad \text{for } t \leq \tau.$$

*Proof.* To prove lemma it is sufficient to show that whenever there exists  $\tau \in (-\infty, t_0]$  such that

$$(4.167) \quad u(\tau) = \inf \{ u(t) : t \leq t_0 \}$$

then  $u$  satisfies (4.166), and so (4.165) holds necessarily. Therefore, let  $\tau \in (-\infty, t_0]$  be arbitrary but fixed such that (4.167) holds. Put

$$(4.168) \quad z(t) = u(t) - u(\tau) \quad \text{for } t \leq \tau.$$

Then, in view of (2.26), (4.164), (4.167), and (4.168), we have

$$(4.169) \quad z(t) \geq 0 \quad \text{for } t \leq \tau,$$

$$(4.170) \quad z'(t) \geq \ell_0(1)(t)u(\tau) - \ell_1(u)(t) \geq -\ell_1(z)(t) \quad \text{for a. e. } t \leq \tau, \quad z(\tau) = 0.$$

Now, applying Lemma 4.23, on account of the inclusion  $\ell_1 \in V_{t_0}$ , (4.170) yields

$$(4.171) \quad z(t) \leq 0 \quad \text{for } t \leq \tau.$$

Thus (4.168), (4.169), and (4.171) implies (4.166).  $\square$

**Lemma 4.27.** *Let  $\ell_0, \ell_1 \in V_{t_0}$ ,  $\omega \in \Sigma$ , (2.26) and (2.30) hold, and let there exist a function  $\gamma \in AC_{loc}((-\infty, t_0]; (0, +\infty))$  satisfying (2.6). Let, moreover,  $u \in AC_{loc}((-\infty, t_0]; \mathbb{R}_+)$  satisfy (4.164), and let there exist a finite limit  $u(-\infty)$ . Then*

$$(4.172) \quad u(-\infty) \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t [\ell_0(1)(s) - \ell_1(1)(s)] ds + \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t q(s) ds = 0$$

*where*

$$(4.173) \quad q(t) \stackrel{\text{def}}{=} u'(t) - \ell_0(u)(t) + \ell_1(u)(t) \quad \text{for a. e. } t \leq t_0.$$

*Proof.* Assume on the contrary that (4.172) does not hold. Then, in view of (2.26), (4.164), and (4.173) we have

$$u(-\infty) \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t [\ell_0(1)(s) - \ell_1(1)(s)] ds + \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t q(s) ds > 0.$$

Therefore, according to (2.30), there exists  $\delta > 0$  such that

$$(4.174) \quad u(-\infty) \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t [\ell_0(1)(s) - \ell_1(1)(s)] ds + \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t q(s) ds > 2\delta M_\omega$$

where

$$M_\omega = \sup \left\{ \int_{\omega(t)}^t \ell_1(1)(s) ds : t \leq t_0 \right\}.$$

Further, note that according to Lemma 4.26, the inequality (4.165) holds and, moreover, there exists  $t_\delta \leq t_0$  such that

$$(4.175) \quad u(t) \leq u(-\infty) + \delta \quad \text{for } t \leq t_\delta.$$

Now the integration of (4.173) from  $\omega(t)$  to  $t$  yields

$$u(t) - u(\omega(t)) = \int_{\omega(t)}^t [\ell_0(u)(s) - \ell_1(u)(s) + q(s)] ds \quad \text{for } t \leq t_\delta,$$

whence, in view of (2.30), (4.165), and (4.175), we get

$$(4.176) \quad u(t) - u(\omega(t)) \geq u(-\infty) \int_{\omega(t)}^t [\ell_0(1)(s) - \ell_1(1)(s)] ds - \delta M_\omega + \int_{\omega(t)}^t q(s) ds \quad \text{for } t \leq t_\delta.$$

Now (4.176), on account of (2.26), (4.164), and (4.173), results in

$$\begin{aligned} \delta M_\omega &\geq u(-\infty) \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t [\ell_0(1)(s) - \ell_1(1)(s)] ds, \\ \delta M_\omega &\geq \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t q(s) ds. \end{aligned}$$

However, the latter inequalities contradict (4.174).  $\square$

## 5. PROOFS

*Proof of Theorem 2.1.* Let  $(a_n)_{n=1}^{+\infty}$  be a sequence of real numbers satisfying (4.93), and let  $n \in \mathbb{N}$  be arbitrary but fixed. Then, according to Lemma 4.8 we have

$$-\ell_1 \in \mathcal{S}_{a_n t_0}(t_0).$$

Moreover, according to Lemma 4.1 with  $\ell = \ell_1$ ,  $\alpha = \vartheta(\gamma)$ ,  $\beta = \vartheta(1/\gamma)$ , and  $\vartheta$  given by (1.6), from (2.7) it follows that (2.26) is fulfilled. Finally, according to (4.91) and (4.92), there exists  $q_0 \in L([a_n, t_0]; \mathbb{R}_+)$  such that

$$|\bar{f}(v)(t)| \leq q_0(t) \quad \text{for a. e. } t \in [a_n, t_0], \quad v \in C([a_n, t_0]; \mathbb{R})$$

and, with respect to (2.2) and (2.3), we have

$$(5.1) \quad \bar{f}(v)(t) \geq 0 \quad \text{for a. e. } t \in [a_n, t_0], \quad v \in C([a_n, t_0]; \mathbb{R}),$$

$$(5.2) \quad \bar{f}(v)(t) = 0 \quad \text{for a. e. } t \in [a_n, t_0], \quad -v \in C([a_n, t_0]; \mathbb{R}_+).$$

Consequently, all the assumptions of Lemma 4.14 (with  $f = \bar{f}$ ,  $a = a_n$ ,  $q = q_0$ ) are fulfilled. Therefore, there exists a solution  $u_n$  to the problem (4.94), (1.2) on  $[a_n, t_0]$  satisfying

$$(5.3) \quad 0 \leq u_n(t) \quad \text{for } t \in [a_n, t_0].$$

Furthermore, according to (5.1), from (4.94) we obtain

$$u_n'(t) \geq \ell_0(u_n)(t) - \ell_1(u_n)(t) \quad \text{for a. e. } t \in [a_n, t_0].$$

Thus, all the assumptions of Lemma 4.9 (with  $a = a_n$ ) are fulfilled, and so

$$(5.4) \quad u_n(t) \leq \kappa \quad \text{for } t \in [a_n, t_0].$$

Now, (5.3) and (5.4) imply (4.95). Consequently, the theorem follows from Lemmas 4.17 and 4.19.  $\square$

*Proof of Theorem 2.2.* Let  $(a_n)_{n=1}^{+\infty}$  be a sequence of real numbers satisfying (4.93), and let  $n \in \mathbb{N}$  be arbitrary but fixed. Then, according to (4.91) and (4.92), with respect to (2.2) and (2.3), we have (5.1) and (5.2). Consequently, all the assumptions of Lemma 4.15 (with  $f = \bar{f}$ ,  $a = a_n$ ) are fulfilled. Therefore, there exists a solution  $u_n$  to the problem (4.94), (1.2) on  $[a_n, t_0]$  satisfying (5.3) and

$$(5.5) \quad u'_n(t) \geq 0 \quad \text{for a. e. } t \in [a_n, t_0].$$

Thus, (1.2), (5.3), and (5.5), with respect to  $c \in [0, \kappa]$ , imply (4.95). Consequently, the theorem follows from Lemmas 4.17 and 4.19.  $\square$

*Proof of Theorem 2.3.* According to Theorem 2.1, there exists a global solution  $u$  to the problem (1.1), (1.2) satisfying (2.10). We will show that  $u$  is positive in  $(-\infty, t_0]$ . Assume on the contrary that there exists  $\tau < t_0$  such that  $u(\tau) = 0$ . Then, in view of  $\ell_0, f \in V_{t_0}$ , (2.3), and (2.10), from (1.1) we obtain (4.139).

On the other hand, in view of the inclusion  $\ell_1 \in V_{t_0}$ , from (2.6) it follows that (4.137) holds. Therefore, according to Lemma 4.23, on account of (2.10) we get (4.120). Now Lemma 4.20 yields that  $u \equiv 0$  on  $(-\infty, t_0]$  which, together with  $c > 0$ , contradicts (1.2).  $\square$

*Proof of Theorem 2.4.* According to Theorem 2.2, there exists a global solution  $u$  to the problem (1.1), (1.2) satisfying (2.10) and (2.11). We will show that  $u$  is positive in  $(-\infty, t_0]$ . Assume on the contrary that there exists  $\tau < t_0$  such that  $u(\tau) = 0$ . Then from (2.10) and (2.11) we get (4.120). Now Lemma 4.20 yields that  $u \equiv 0$  on  $(-\infty, t_0]$  which, together with  $c > 0$ , contradicts (1.2).  $\square$

*Proof of Theorem 2.5.* First note that, according to the inclusion  $\ell_1 \in V$ , from (2.16) it follows that (2.6) holds. Therefore, according to Theorem 2.1, there exists a global solution  $u$  to the problem (1.1), (1.2) satisfying (2.10). We will show that  $u$  is positive in  $[t_0, +\infty)$ . Assume on the contrary that (2.17) does not hold. Then, in view of (1.2), there exists  $\tau > t_0$  such that  $u(\tau) = 0$  and

$$(5.6) \quad u(t) > 0 \quad \text{for } t \in [t_0, \tau).$$

Now, on account of (2.10), (2.15), (5.6), and the inclusions  $\ell_0, f \in V$ , from (1.1) we obtain (4.139). Moreover, from (2.16), with respect to the inclusion  $\ell_1 \in V$ , the inequality (4.137) follows. Therefore, according to Lemma 4.23 we have (4.152). However, (4.152) contradicts (5.6).  $\square$

*Proof of Theorem 2.6.* Put

$$(5.7) \quad \tilde{f}(v)(t) \stackrel{\text{def}}{=} f(|v|)(t) \quad \text{for a. e. } t \in \mathbb{R}, \quad v \in C_{loc}(\mathbb{R}; \mathbb{R})$$

and consider the auxiliary equation

$$(5.8) \quad u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \tilde{f}(u)(t).$$

Note that from (2.18) it follows that

$$\tilde{f}(v)(t) \operatorname{sgn} v(t) \leq q(t, \|v\|) \quad \text{for a. e. } t \geq t_0, \quad v \in C_0(\mathbb{R}; \mathbb{R}), \quad -\kappa \leq v(t) \leq \kappa \quad \text{for } t \leq t_0$$

holds. Therefore, according to Theorem 2.2, for every  $c \in [0, \kappa]$  there exists a global solution  $u$  to the problem (5.8), (1.2) satisfying (2.10) and (2.11).



Put

$$(5.9) \quad w(t) = \sup \{u(s) : s \leq t\} \quad \text{for } t \in \mathbb{R}$$

and

$$A = \{t \in \mathbb{R} : w(t) = u(t)\}.$$

Obviously, on account of (5.9) and (2.11) we have

$$(5.10) \quad w \in AC_{loc}(\mathbb{R}; \mathbb{R}),$$

$$(5.11) \quad w(t) \geq 0 \quad \text{for } t \in \mathbb{R},$$

$$(5.12) \quad w'(t) \geq 0 \quad \text{for a. e. } t \in \mathbb{R},$$

$$(5.13) \quad w(t) = u(t) \quad \text{for } t \leq t_0,$$

$$(5.14) \quad w(t) \geq u(t) \quad \text{for } t \geq t_0,$$

and

$$(5.15) \quad w'(t) = \begin{cases} u'(t) & \text{for a. e. } t \in A, \\ 0 & \text{for a. e. } t \in \mathbb{R} \setminus A. \end{cases}$$

According to (5.13) and (5.14), from (5.8) it follows that

$$(5.16) \quad u'(t) \leq \ell_0(w)(t) - \ell_1(u)(t) + \tilde{f}(u)(t) \quad \text{for a. e. } t \in \mathbb{R}.$$

On the other hand, in view of the inclusions  $\ell_0 - \ell_1 \in \mathcal{P}^+$  and  $f \in V$ , on account of (2.18), (5.7), and (5.10)–(5.14), we have

$$(5.17) \quad \ell_0(w)(t) - \ell_1(u)(t) + \tilde{f}(u)(t) \geq \ell_0(w)(t) - \ell_1(w)(t) \geq 0 \quad \text{for a. e. } t \in \mathbb{R}.$$

Now from (5.15)–(5.17) we get

$$(5.18) \quad w'(t) \leq \ell_0(w)(t) - \ell_1(u)(t) + \tilde{f}(u)(t) \quad \text{for a. e. } t \in \mathbb{R}.$$

Put

$$(5.19) \quad z(t) = w(t) - u(t) \quad \text{for } t \in \mathbb{R}.$$

Then, in view of (5.8), (5.13), (5.18), and (5.19), we have

$$(5.20) \quad z'(t) \leq \ell_0(z)(t) \quad \text{for a. e. } t \in \mathbb{R},$$

$$(5.21) \quad z(t) = 0 \quad \text{for } t \leq t_0.$$

Now the inclusion  $\ell_0 \in V$ , according to Proposition 4.1, yields  $\ell_0 \in \mathcal{S}_{t_0\tau}(t_0)$  for every  $\tau > t_0$ . Consequently, (5.20) and (5.21) result in  $z(t) \leq 0$  for  $t \geq t_0$  whence, on account of (5.19), we get

$$(5.22) \quad w(t) \leq u(t) \quad \text{for } t \geq t_0.$$

However, (5.13), (5.14), and (5.22) yield that  $w \equiv u$  on  $\mathbb{R}$ , and, consequently, on account of (5.7), (5.8), (5.11), and (5.12), we have that  $u$  is a global solution also to the problem (1.1), (1.2) satisfying (2.10) and (2.19).  $\square$

*Proof of Theorem 2.7.* Let  $u$  be a solution to (1.1) on  $(-\infty, t_0]$  satisfying (2.10), and let  $\tau \in \mathbb{R}$ ,  $\tau \leq t_0$ , be arbitrary but fixed. Then, according to (2.21),  $u$  satisfies also (4.138) and (4.139). Moreover, since  $\ell_1 \in V_{t_0}$ , the inequality (4.137) holds. Thus, according to Lemma 4.21 we have (4.140), and so the function  $u/\gamma$  is nondecreasing in  $(-\infty, t_0]$ . Consequently, in view of (2.10), there exists a finite limit

$$(5.23) \quad 0 \leq \lim_{t \rightarrow -\infty} \frac{u(t)}{\gamma(t)} < +\infty.$$

Now from (2.22) and (5.23) it follows that there exists a finite limit  $u(-\infty)$ .  $\square$

*Proof of Theorem 2.8.* Let  $u$  be a solution to (1.1) on  $(-\infty, t_0]$  satisfying (2.10). Then, in view of (2.3),  $u$  satisfies also (4.148). Thus, according to Lemma 4.22, the estimate (4.150) holds. Therefore, in view of (2.3) and (2.10), we have (4.164) whence, on account of (4.150) we get

$$(5.24) \quad u'(t) \geq \ell_0(u)(t) - \frac{\ell_1(\gamma)(t)}{\gamma(t)} u(t) \quad \text{for a. e. } t \leq t_0.$$

Now, (2.1), (2.7), (2.8), (2.10), and (5.24) yield that all the assumptions of Lemma 4.25 are fulfilled with  $p = \ell_1(\gamma)/\gamma$ . Therefore, there exists a finite limit  $u(-\infty)$ .  $\square$

*Proof of Theorem 2.9.* First note that according to Remark 2.9 we have (2.23). Further, (2.3) and (2.10) yields (4.164). Therefore, according to Lemma 4.27, on account of (2.23), we have

$$(5.25) \quad u(-\infty) \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t \ell_0(1)(s) ds + \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t f(u)(s) ds = 0$$

for any  $\omega \in \Sigma$ .

Consequently, if (2.27) holds then we define  $\omega$  in the following way: let the values  $\omega(t_n)$  and  $\omega(t_0)$  be defined by

$$(5.26) \quad \int_{\omega(t_{n-1})}^{t_{n-1}} \ell_0(1)(s) ds = 1 \quad \text{for } n \in \mathbb{N},$$

where

$$(5.27) \quad t_n = t_0 - n \quad \text{for } n \in \mathbb{N}.$$

Further, put

$$\omega(t) = (\omega(t_{n-1}) - \omega(t_n))(t - t_n) + \omega(t_n) \quad \text{for } t \in (t_n, t_{n-1}), \quad n \in \mathbb{N}$$

and  $\omega(t) \stackrel{\text{def}}{=} \omega(t_0)$  for  $t > t_0$ . Then, obviously,  $\omega \in \Sigma$  and, in view of (5.26) and (5.27), we have

$$\limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t \ell_0(1)(s) ds > 0.$$

Thus, in view of (2.3) and (2.10), from (5.25) it follows that (2.29) is fulfilled.

Further note that, in view of Lemma 4.26 we have (4.165) and if  $c = u(-\infty)$  then  $u(t) = c$  for  $t \leq t_0$ . Therefore, if (2.28) holds then, on account of (2.10), we get (2.29) again.  $\square$

*Proof of Theorem 2.10.* First note that (2.3) and (2.10) yields (4.164). Therefore, according to Lemma 4.27, we have

$$(5.28) \quad u(-\infty) \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t [\ell_0(1)(s) - \ell_1(1)(s)] ds + \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t f(u)(s) ds = 0.$$

Consequently, if (2.31) holds, then, in view of (2.3) and (2.10), from (5.28) it follows that (2.29) is fulfilled.

Further note that, in view of Lemma 4.26 we have (4.165) and if  $c = u(-\infty)$  then  $u(t) = c$  for  $t \leq t_0$ . Therefore, if (2.28) holds then, on account of (2.10), we get (2.29) again.  $\square$

*Proof of Proposition 2.1.* Let  $c \in (0, \kappa)$  be arbitrary but fixed and let  $u \in C_0(\mathbb{R}; [0, \kappa])$  satisfy conditions of Definition 2.1 with  $\tau = t_0$ . We will show that

$$(5.29) \quad \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t f(u)(s) ds > 0.$$

Obviously, for every  $n \in \mathbb{N}$  there exists  $t_n \leq t_0$  such that

$$u(-\infty) \leq u(t) \leq u(-\infty) + \frac{c - u(-\infty)}{n} \quad \text{for } t \leq t_n.$$

Put

$$\vartheta_n(u)(t) = \begin{cases} u(t) & \text{if } t \leq t_n, \\ u(t_n) & \text{if } t > t_n \end{cases} \quad \text{for } n \in \mathbb{N}.$$

Then

$$(5.30) \quad 0 < \vartheta_n(u)(t) \leq c \quad \text{for } t \in \mathbb{R}, \quad n \in \mathbb{N},$$

$$(5.31) \quad \|\vartheta_n(u) - u(-\infty)\| \leq \frac{c - u(-\infty)}{n} \quad \text{for } n \in \mathbb{N},$$

and, on account of the inclusion  $f \in V_{t_0}$ , for every  $n \in \mathbb{N}$  we have

$$(5.32) \quad \int_{\omega(t)}^t f(u)(s)ds = \int_{\omega(t)}^t f(\vartheta_n(u))(s)ds \quad \text{for } t \leq t_n.$$

On the other hand, in view of (5.31) and the continuity of  $h_1$ , there exist  $\varepsilon_n > 0$  ( $n \in \mathbb{N}$ ) such that

$$(5.33) \quad \lim_{n \rightarrow +\infty} \varepsilon_n = 0$$

and

$$(5.34) \quad \|h_1(\vartheta_n(u)) - h_1(u(-\infty))\|_\infty \leq \varepsilon_n \quad \text{for } n \in \mathbb{N}.$$

Now, (2.32), (5.30), (5.32), and (5.34) results in

$$(5.35) \quad \int_{\omega(t)}^t f(u)(s)ds \geq \int_{\omega(t)}^t g(s)h_1(u(-\infty))(s)ds - \varepsilon_n g^* \quad \text{for } t \leq t_n, \quad n \in \mathbb{N}$$

where

$$(5.36) \quad g^* = \sup \left\{ \int_{\omega(t)}^t |g(s)|ds : t \leq t_0 \right\}.$$

Consequently, from (5.35) it follows that

$$(5.37) \quad \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t f(u)(s)ds \geq \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t g(s)h_1(u(-\infty))(s)ds - \varepsilon_n g^* \quad \text{for } n \in \mathbb{N}.$$

Thus (5.37), on account of (2.33), (2.34), (5.33), and (5.36), yields (5.29).  $\square$

*Proof of Corollary 2.3.* Let  $u$  be a solution to the problem (1.1), (1.2) on  $(-\infty, t_0]$  satisfying (2.10). Then, in view of (2.32) and (2.35), all the conditions of Theorem 2.7 are satisfied. Therefore, there exists a finite limit  $u(-\infty)$ .

Now we will show that (2.36) and (2.37) imply (2.33) and (2.34) with a suitable function  $\omega$ . Let

$$(5.38) \quad \varphi(t) = \frac{1}{(t_0 + 1 - t)^2} \quad \text{for } t \leq t_0.$$

Then, obviously,

$$(5.39) \quad \varphi(t) > 0 \quad \text{for } t \leq t_0, \quad \lim_{t \rightarrow -\infty} \int_t^{t_0} \varphi(s)ds = 1.$$

Define  $\omega$  by

$$(5.40) \quad \int_{\omega(t)}^{t_0} (g(s) + \varphi(s)) ds = 1 + \int_t^{t_0} (g(s) + \varphi(s)) ds \quad \text{for } t \leq t_0$$

and  $\omega(t) \stackrel{\text{def}}{=} \omega(t_0)$  for  $t > t_0$ . Then, in view of (5.39) and the non-negativity of  $g$ , we have  $\omega \in \Sigma$ . Moreover, (5.39) and (5.40) yields

$$(5.41) \quad \lim_{t \rightarrow -\infty} \int_{\omega(t)}^t \varphi(s) ds = 0,$$

$$(5.42) \quad \int_{\omega(t)}^t g(s) ds = 1 - \int_{\omega(t)}^t \varphi(s) ds \quad \text{for } t \leq t_0.$$

Therefore, from (5.41) and (5.42) we get (2.33) and

$$(5.43) \quad \lim_{t \rightarrow -\infty} \int_{\omega(t)}^t g(s) ds = 1.$$

Now (2.34) follows from (2.37) and (5.43).

Consequently, according to Proposition 2.1, all the assumptions of Theorem 2.9 hold and so (2.29) is satisfied.  $\square$

*Proof of Corollary 2.4.* Let  $u$  be a solution to the problem (1.1), (1.2) on  $(-\infty, t_0]$  satisfying (2.10). Then, in view of (2.32) and (2.35), all the conditions of Theorem 2.8 are satisfied. Therefore, there exists a finite limit  $u(-\infty)$ .

Further, (2.1) implies  $\ell_0 \in V_{t_0}$ , (2.6) implies that  $\gamma$  is a nonincreasing function, and so, according to Lemma 4.1 with  $\ell = \ell_1$ ,  $\alpha = \vartheta(\gamma)$ ,  $\beta = \vartheta(1/\gamma)$ , and  $\vartheta$  given by (1.6), from (2.7) we get (2.26). Finally, according to Proposition 2.1, the inclusion (2.28) holds. Consequently, all the assumptions of Theorem 2.10 are fulfilled and so (2.29) is satisfied.  $\square$

*Proof of Theorem 3.1.* Define operators  $\ell_0$ ,  $\ell_1$ , and  $f$  by

$$(5.44) \quad \ell_i(u)(t) = p_i(t)u(\mu_i(t)) \quad \text{for a. e. } t \in \mathbb{R} \quad (i = 0, 1),$$

$$(5.45) \quad f(u)(t) = h(t, u(t), u(\nu(t))) \quad \text{for a. e. } t \in \mathbb{R}.$$

Then, in view of (3.4) we have  $\ell_0, \ell_1, f \in V$ , and so from (3.1) and (3.3) it follows that (2.2) and (2.3) hold.

Further, put

$$(5.46) \quad \gamma(t) = \exp \left( e \int_t^{t_0} p_1(s) ds \right) \quad \text{for } t \leq t_0.$$

Then, in view of (3.5), we have

$$(5.47) \quad \gamma(t) = \gamma(\mu_1(t)) \exp \left( -e \int_{\mu_1(t)}^t p_1(s) ds \right) \geq \frac{\gamma(\mu_1(t))}{e} \quad \text{for a. e. } t \leq t_0,$$

and so

$$(5.48) \quad \gamma'(t) = -ep_1(t)\gamma(t) \leq -p_1(t)\gamma(\mu_1(t)) \quad \text{for a. e. } t \leq t_0.$$

Consequently, (2.6) holds. Moreover, from (3.6), on account of (5.46), we get (2.7). Finally, from (3.7), in view of (3.5), we obtain

$$(5.49) \quad M_\mu < +\infty.$$

Now, let  $c \in [0, \kappa e^{-M_\mu})$ . Then there exists  $\varepsilon > 0$  such that

$$(5.50) \quad c \leq \kappa e^{-(M_\mu + \varepsilon)}.$$

Put

$$\varphi(t) = \frac{\varepsilon}{(t_0 + 1 - t)^2} \quad \text{for } t \leq t_0.$$

Then, obviously,

$$(5.51) \quad \varphi(t) > 0 \quad \text{for } t \leq t_0, \quad \lim_{t \rightarrow -\infty} \int_t^{t_0} \varphi(s) ds = \varepsilon.$$

Define a function  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  by the equalities

$$(5.52) \quad \int_{\sigma(t)}^{t_0} (P_1(s) + \varphi(s)) ds = M_\mu + \varepsilon + \int_t^{t_0} (P_1(s) + \varphi(s)) ds \quad \text{for } t \leq t_0$$

and  $\sigma(t) \stackrel{\text{def}}{=} \sigma(t_0)$  for  $t > t_0$ , where

$$(5.53) \quad P_1(t) = p_1(t) \exp \left( e \int_{\mu_1(t)}^t p_1(s) ds \right) \quad \text{for a. e. } t \leq t_0.$$

Then, in view of (5.51), (5.53), and the non-negativity of  $p_1$ , we have  $\sigma \in \Sigma$ . Moreover, (5.49) and (5.51)–(5.53) yield

$$(5.54) \quad \int_{\sigma(t)}^t p_1(s) \exp \left( e \int_{\mu_1(s)}^s p_1(\xi) d\xi \right) ds = M_\mu + \varepsilon - \int_{\sigma(t)}^t \varphi(s) ds \quad \text{for } t \leq t_0$$

and

$$(5.55) \quad \sup \left\{ \int_{\sigma(t)}^t p_1(s) \exp \left( e \int_{\mu_1(s)}^s p_1(\xi) d\xi \right) ds : t \leq t_0 \right\} = M_\mu + \varepsilon < +\infty.$$

Now, from (3.8) and (5.54), on account of (5.51), we get

$$\int_{\sigma(t)}^t p_1(s) \exp \left( e \int_{\mu_1(s)}^s p_1(\xi) d\xi \right) ds > \int_{\mu_0(t)}^t p_1(s) \exp \left( e \int_{\mu_1(s)}^s p_1(\xi) d\xi \right) ds \quad \text{for a. e. } t \leq t_0$$

whence, in view of the non-negativity of  $p_1$ , we obtain

$$(5.56) \quad \sigma(t) \leq \mu_0(t) \quad \text{for a. e. } t \leq t_0.$$

Therefore, (5.55) and (5.56), with respect to (3.4), (5.46), and (5.50), imply (2.1), (2.8), and  $c \in [0, \kappa e^{-M_\sigma}]$  with  $M_\sigma$  defined by (2.9).

Thus, the assertion follows from Theorem 2.1.  $\square$

*Proof of Theorem 3.2.* Define operators  $\ell_0$ ,  $\ell_1$ , and  $f$  by (5.44) and (5.45). Then, in view of (3.4) we have  $\ell_0, \ell_1, f \in V$ , and so from (3.1) and (3.3) it follows that (2.2) and (2.3) hold. We will show that (3.11) and (3.12) imply  $\ell_0 - \ell_1 \in \mathcal{P}_{t_0}^+$ . Indeed, let  $u \in AC_{loc}((-\infty, t_0]; \mathbb{R}_+)$  be a nondecreasing function. Then, in view of (3.12), we have

$$(5.57) \quad p_1(t)(u(\mu_0(t)) - u(\mu_1(t))) \geq 0 \quad \text{for a. e. } t \leq t_0.$$

On the other hand, on account of (5.44), we find

$$(5.58) \quad \ell_0(u)(t) - \ell_1(u)(t) = (p_0(t) - p_1(t))u(\mu_0(t)) + p_1(t)(u(\mu_0(t)) - u(\mu_1(t))) \quad \text{for a. e. } t \leq t_0.$$

Thus, from (5.58), in view of (3.11) and (5.57), we obtain

$$\ell_0(u)(t) - \ell_1(u)(t) \geq 0 \quad \text{for a. e. } t \leq t_0.$$

Consequently,  $\ell_0 - \ell_1 \in \mathcal{P}_{t_0}^+$ , and the assertion follows from Theorem 2.2.  $\square$

*Proof of Theorem 3.3.* Let  $c \in (0, \kappa e^{-M_\mu})$  be arbitrary but fixed. According to Theorem 3.1, there exists a global solution  $u$  to the problem (1.5), (1.2) satisfying (2.10). We will show that  $u$  is positive in  $(-\infty, t_0]$ . Assume on the contrary that there exists  $\tau < t_0$  such that  $u(\tau) = 0$ . Define operators  $\ell_0, \ell_1$ , and  $f$  by (5.44) and (5.45). Then, in view of (3.4) we have  $\ell_0, \ell_1, f \in V$  and (4.139) is fulfilled.

Further, define  $\gamma$  by (5.46). Then, in view of (3.5), we have (5.47) and (5.48). Consequently, (4.137) holds. Therefore, according to Lemma 4.23, on account of (2.10), we get (4.120). Now Lemma 4.20 yields that  $u \equiv 0$  on  $(-\infty, t_0]$  which, together with  $c > 0$ , contradicts (1.2).  $\square$

*Proof of Theorem 3.4.* Let  $c \in (0, \kappa e^{-M_\mu})$  be arbitrary but fixed. According to Theorem 3.2, there exists a global solution  $u$  to the problem (1.5), (1.2) satisfying (2.10) and (2.11). We will show that  $u$  is positive in  $(-\infty, t_0]$ . Assume on the contrary that there exists  $\tau < t_0$  such that  $u(\tau) = 0$ . Then from (2.10) and (2.11) we get (4.120). Define operators  $\ell_0, \ell_1$ , and  $f$  by (5.44) and (5.45). Then, in view of (3.4) we have  $\ell_0, \ell_1, f \in V$ . Now Lemma 4.20 yields that  $u \equiv 0$  on  $(-\infty, t_0]$  which, together with  $c > 0$ , contradicts (1.2).  $\square$

*Proof of Theorem 3.5.* Analogously to the proof of Theorem 3.1 one can show that all the assumptions of Theorem 2.5 are fulfilled.  $\square$

*Proof of Theorem 3.6.* Analogously to the proof of Theorem 3.2 one can show that all the assumptions of Theorem 2.6 are fulfilled.  $\square$

*Proof of Theorem 3.7.* Define operators  $\ell_0, \ell_1$  and  $f$  by (5.44) and (5.45). Then, in view of (3.19) we have  $\ell_0, \ell_1, f \in V_{t_0}$ , and so from (3.1) it follows that (2.21) holds.

Furthermore, define  $\gamma$  by (5.46). Then, on account of (3.5), we have (5.47) and (5.48). Therefore, (2.6) holds and (3.20) implies (2.22). Consequently, the assertion follows from Theorem 2.7.  $\square$

*Proof of Theorem 3.8.* Define operators  $\ell_0, \ell_1$ , and  $f$  by (5.44) and (5.45). Then, in view of (3.19) we have  $\ell_0, \ell_1, f \in V_{t_0}$ , and so from (3.1) it follows that (2.3) holds.

Further, define  $\gamma$  by (5.46). Then, in view of (3.5), we have (5.47) and (5.48). Consequently, (2.6) holds. Moreover, from (3.6), on account of (5.46), we get (2.7).

Now, let

$$(5.59) \quad p^* = \text{ess sup} \left\{ \int_{\mu_0(t)}^t p_1(s) ds : t \leq t_0 \right\},$$

and let  $\varphi$  be given by (5.38). Then (5.39) holds. Define a function  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  by the equalities

$$(5.60) \quad \int_{\sigma(t)}^{t_0} (p_1(s) + \varphi(s)) ds = p^* + 1 + \int_t^{t_0} (p_1(s) + \varphi(s)) ds \quad \text{for } t \leq t_0$$

and  $\sigma(t) \stackrel{\text{def}}{=} \sigma(t_0)$  for  $t > t_0$ . Then, in view of (5.39) and the non-negativity of  $p_1$ , we have  $\sigma \in \Sigma$ . Moreover, (5.39) and (5.60) yields

$$(5.61) \quad \int_{\sigma(t)}^t p_1(s) ds = p^* + 1 - \int_{\sigma(t)}^t \varphi(s) ds > p^* \quad \text{for } t \leq t_0$$

and

$$(5.62) \quad \sup \left\{ \int_{\sigma(t)}^t p_1(s) ds : t \leq t_0 \right\} < +\infty.$$

Now, from (5.59) and (5.61) we get

$$\int_{\sigma(t)}^t p_1(s)ds > \int_{\mu_0(t)}^t p_1(s)ds \quad \text{for a. e. } t \leq t_0$$

whence, in view of non-negativity of  $p_1$ , we obtain (5.56). Therefore, (5.56) and (5.62), with respect to (3.19), (3.5), and (5.46), implies (2.1) and (2.8).

Thus, the assertion follows from Theorem 2.8.  $\square$

*Proof of Theorem 3.9.* Assume on the contrary that  $u(-\infty) \in (0, \kappa)$ . Then, in view of (3.21), there exist  $\delta > 0$  and  $t_\delta \leq t_0$  such that

$$(5.63) \quad h_1(x, y) > 0 \quad \text{for } x, y \in [u(-\infty) - \delta, u(-\infty) + \delta],$$

$$(5.64) \quad 0 < u(-\infty) - \delta \leq u(t) \leq u(-\infty) + \delta < \kappa \quad \text{for } t \leq t_\delta.$$

Integrating (1.5) from  $t$  to  $t_\delta$  we obtain

$$u(t_\delta) - u(t) = \int_t^{t_\delta} [p_0(s)u(\mu_0(s)) - p_1(s)u(\mu_1(s)) + h(s, u(s), u(\nu(s)))] ds \quad \text{for } t \leq t_\delta$$

whence, on account of (3.19), (3.22), and (5.64) we get

$$(5.65) \quad u(t_\delta) - u(t) \geq (u(-\infty) - \delta) \int_t^{t_\delta} p_0(s)ds - \kappa \int_t^{t_\delta} p_1(s)ds \\ + \int_t^{t_\delta} g(s)h_1(u(s), u(\nu(s)))ds \quad \text{for } t \leq t_\delta.$$

On the other hand, in view of (3.19), (5.63), and (5.64) we have

$$(5.66) \quad h_1(u(t), u(\nu(t))) \geq h_* > 0 \quad \text{for a. e. } t \leq t_\delta$$

where

$$(5.67) \quad h_* = \min \{h_1(x, y) : u(-\infty) - \delta \leq x, y \leq u(-\infty) + \delta\}.$$

Therefore, if (2.36) or (3.23) is fulfilled then, on account of (3.20), (5.64), and (5.66), from (5.65) we obtain  $u(-\infty) = -\infty$ , a contradiction.  $\square$

*Proof of Theorem 3.10.* Assume on the contrary that  $u(-\infty) \in (0, \kappa)$ . Then, in view of (3.21), there exist  $\delta > 0$  and  $t_\delta \leq t_0$  such that (5.63) and (5.64) hold. Therefore, on account of (3.19), (5.63), and (5.64), we have (5.66) where  $h_*$  is given by (5.67). Moreover, from (1.5), with respect to (3.19), (3.22), and (5.66), we get

$$(5.68) \quad u'(t) \geq p_0(t)u(\mu_0(t)) - p_1(t)u(\mu_1(t)) \quad \text{for a. e. } t \leq t_\delta.$$

Define operators  $\ell_0$  and  $\ell_1$  by (5.44). Then, in view of (3.19), (3.11), (3.25), and (5.68) we have  $\ell_0, \ell_1 \in V_{t_\delta}$ ,

$$\ell_0(1)(t) \geq \ell_1(1)(t) \quad \text{for a. e. } t \leq t_\delta, \\ \sup \left\{ \int_{\omega(t)}^t \ell_1(1)(s)ds : t \leq t_\delta \right\} < +\infty, \\ u'(t) \geq \ell_0(u)(t) - \ell_1(u)(t) \quad \text{for a. e. } t \leq t_\delta.$$

Further, define  $\gamma$  by (5.46). Then, on account of (3.5), we have (5.47) and (5.48). Consequently,

$$\gamma'(t) \leq -\ell_1(\gamma)(t) \quad \text{for a. e. } t \leq t_\delta,$$

Therefore, all the assumptions of Lemma 4.27 (with  $t_0 = t_\delta$ ) are fulfilled, and thus

$$(5.69) \quad u(-\infty) \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t [p_0(s) - p_1(s)] ds + \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t h(s, u(s), u(\nu(s))) ds = 0.$$

However, from (3.22), (5.66), and (5.69), with respect to (3.11), the inclusions  $u(-\infty) \in (0, \kappa)$ ,  $\omega \in \Sigma$ , and the non-negativity of  $g$ , it follows that

$$(5.70) \quad u(-\infty) \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t [p_0(s) - p_1(s)] ds + h_* \limsup_{t \rightarrow -\infty} \int_{\omega(t)}^t g(s) ds = 0.$$

Now it is clear that each of the conditions (3.26) and (3.27) contradicts (5.70).  $\square$

*Proof of Corollary 3.3.* The assertion follows from Theorems 3.7 and 3.9.  $\square$

*Proof of Corollary 3.4.* The assertion follows from Corollary 3.3 and Lemma 4.26 with  $\ell_i$  ( $i = 0, 1$ ) and  $\gamma$  defined by (5.44) and (5.46), respectively.  $\square$

*Proof of Corollary 3.5.* Note that from (3.6), on account of (3.19), we have (3.11). Therefore, the assertion follows from Theorems 3.8 and 3.10 with  $\omega(t) = t - 1$  for  $t \in \mathbb{R}$ , and Lemma 4.26 with  $\ell_i$  ( $i = 0, 1$ ) and  $\gamma$  defined by (5.44) and (5.46), respectively.  $\square$

## 6. APPLICATIONS

In this section we apply the results obtained above to the model equations appearing in natural sciences.

**Generalized logistic equation:** Consider the generalized logistic equation

$$(6.1) \quad u'(t) = g_0(t)u(t) \int_{\nu(t)}^t \left| 1 - \frac{u(s)}{\kappa} \right|^\lambda \operatorname{sgn} \left( 1 - \frac{u(s)}{\kappa} \right) d_s K(t, s),$$

where  $g_0 \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$ ,  $\nu : \mathbb{R} \rightarrow \mathbb{R}$  is a locally essentially bounded function,  $\nu(t) \leq t$  for almost every  $t \in \mathbb{R}$ ,  $\kappa > 0$ ,  $\lambda > 0$ , and  $K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function satisfying the following conditions:

- $K(t, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is a left continuous nondecreasing function of locally bounded variation for almost every  $t \in \mathbb{R}$ ,
- $\mathcal{K} : \mathbb{R} \rightarrow \mathbb{R}$ , where

$$\mathcal{K}(t) \stackrel{\text{def}}{=} \int_{\nu(t)}^t d_s K(t, s) \quad \text{for a. e. } t \in \mathbb{R},$$

is an essentially bounded measurable function.

**Theorem 6.1.** *Let*

$$\lim_{t \rightarrow -\infty} \int_t^{t_0} g_0(s) ds = +\infty, \quad \lim_{t \rightarrow -\infty} \operatorname{ess\,inf} \{ \mathcal{K}(s) : s \leq t \} > 0.$$

*Then, for every  $t_0 \in \mathbb{R}$  and  $c \in (0, \kappa)$  there exists a positive global solution  $u$  to (6.1) such that*

$$u(t_0) = c, \quad u(t) < \kappa \quad \text{for } t \leq t_0, \quad u'(t) \geq 0 \quad \text{for a. e. } t \leq t_0,$$

*and there exists a limit  $u(-\infty) = 0$ .*

*If, in addition,*

$$(6.2) \quad \lim_{t \rightarrow +\infty} \operatorname{ess\,inf} \{ \nu(s) : s \geq t \} = +\infty$$



and there exists  $\omega \in \Sigma$  such that

$$(6.3) \quad \lim_{t \rightarrow +\infty} \omega(t) = +\infty, \quad \limsup_{t \rightarrow +\infty} \int_{\omega(t)}^t g_0(s) \mathcal{K}(s) ds > 0,$$

then either  $u$  oscillates about  $\kappa$  in the neighbourhood of  $+\infty$  or there exists a limit  $u(+\infty) = \kappa$ .

*Proof.* Let  $t_0 \in \mathbb{R}$  and  $c \in (0, \kappa)$  be arbitrary but fixed. Consider the auxiliary equation

$$(6.4) \quad u'(t) = g_0(t) \chi(t, u(t)) \int_{\nu(t)}^t \left| 1 - \frac{u(s)}{\kappa} \right|^\lambda \operatorname{sgn} \left( 1 - \frac{u(s)}{\kappa} \right) d_s K(t, s),$$

where

$$\chi(t, x) \stackrel{\text{def}}{=} \begin{cases} (|x| + x)/2 & \text{if } x < U(t), \\ U(t) & \text{if } x \geq U(t) \end{cases} \quad \text{for } t \in \mathbb{R}, \quad x \in \mathbb{R}$$

and

$$U(t) \stackrel{\text{def}}{=} \begin{cases} \kappa & \text{for } t \leq t_0, \\ \kappa \exp \left( \int_{t_0}^t g_0(s) \mathcal{K}(s) ds \right) & \text{for } t > t_0. \end{cases}$$

Put  $\ell_i \equiv 0$  ( $i = 0, 1$ ),  $h_0(x) \stackrel{\text{def}}{=} x$  for  $x \in \mathbb{R}_+$ ,

$$f(v)(t) \stackrel{\text{def}}{=} g_0(t) \chi(t, v(t)) \int_{\nu(t)}^t \left| 1 - \frac{v(s)}{\kappa} \right|^\lambda \operatorname{sgn} \left( 1 - \frac{v(s)}{\kappa} \right) d_s K(t, s) \quad \text{for a. e. } t \in \mathbb{R},$$

$$h_1(v)(t) \stackrel{\text{def}}{=} v(t) \int_{\nu(t)}^t \left| 1 - \frac{v(s)}{\kappa} \right|^\lambda \operatorname{sgn} \left( 1 - \frac{v(s)}{\kappa} \right) d_s K(t, s) \quad \text{for a. e. } t \in \mathbb{R}.$$

Then all the assumptions of Theorem 2.4 are fulfilled with

$$q(t, x) \stackrel{\text{def}}{=} g_0(t) U(t) \mathcal{K}(t) \quad \text{for a. e. } t \geq t_0, \quad x \in \mathbb{R}_+, \quad g(t) \stackrel{\text{def}}{=} g_0(t) \mathcal{K}(t) \quad \text{for a. e. } t \in \mathbb{R}.$$

Therefore, there exists a global solution  $u$  to (6.4) satisfying  $u(t_0) = c$ ,  $u'(t) \geq 0$  for a.e.  $t \leq t_0$ , and  $0 < u(t) < \kappa$  for  $t \leq t_0$ . Moreover, also the assumptions of Corollary 2.3 are fulfilled with  $\gamma \equiv 1$  and  $g \equiv g_0$ . Thus  $u(-\infty) = 0$ .

Now we show that  $u$  is positive also on  $(t_0, +\infty)$ . Assume on the contrary that there exists  $\tau_1 > t_0$  such that  $u(\tau_1) = 0$ . Without loss of generality we can assume that  $u(t) > 0$  for  $t < \tau_1$ . Note that  $u$  is bounded on  $(-\infty, \tau_1]$ , and so there exists  $M > 0$  such that  $u(t) \leq M$  for  $t \leq \tau_1$ . Moreover, there exists  $\tau_0 < \tau_1$  such that  $u(t) < \kappa$  for  $t \in [\tau_0, \tau_1]$ . Consequently, from (6.4) we get

$$\ln \frac{u(t)}{u(\tau_0)} \geq - \left| 1 - \frac{M}{\kappa} \right|^\lambda \int_{\tau_0}^t g_0(s) \mathcal{K}(s) ds \quad \text{for } t \in [\tau_0, \tau_1].$$

Now the latter inequality yields

$$\lim_{t \rightarrow \tau_1} \ln \frac{u(t)}{u(\tau_0)} > -\infty$$

which contradicts  $u(\tau_1) = 0$ .

Finally we show that  $u(t) < U(t)$  for  $t \in \mathbb{R}$  which implies that  $u$  is also a solution to (6.1). Assume on the contrary that there exists  $\tau \in \mathbb{R}$  such that  $u(\tau) = U(\tau)$ . Obviously, according

to the above proven,  $\tau > t_0$  and without loss of generality we can assume that  $u(t) < U(t)$  for  $t \in [t_0, \tau)$ . Thus from (6.4) we get

$$\begin{aligned} u(\tau) &= u(t_0) \exp \left( \int_{t_0}^{\tau} g_0(t) \int_{\nu(t)}^t \left| 1 - \frac{u(s)}{\kappa} \right|^{\lambda} \operatorname{sgn} \left( 1 - \frac{u(s)}{\kappa} \right) d_s K(t, s) dt \right) \\ &\leq u(t_0) \exp \left( \int_{t_0}^{\tau} g_0(t) \mathcal{K}(t) dt \right) < U(\tau). \end{aligned}$$

However, the latter inequality contradicts our assumption.

Let, in addition, (6.2) hold and let  $\omega \in \Sigma$  be such that (6.3) is fulfilled. Then either  $u$  oscillates about  $\kappa$  in the neighbourhood of  $+\infty$  or there exists  $\tau \in \mathbb{R}$  such that

$$(6.5) \quad u(t) \leq \kappa \quad \text{for } t \geq \tau$$

or

$$(6.6) \quad u(t) \geq \kappa \quad \text{for } t \geq \tau.$$

From (6.1), in view of (6.2), it follows that  $u$  is eventually nondecreasing if (6.5) holds and eventually nonincreasing if (6.6) is fulfilled. Thus, in both cases there exists a finite limit  $u(+\infty)$ . Therefore, from (6.1) we get

$$(6.7) \quad \ln \frac{u(t)}{u(\omega(t))} \geq \left| 1 - \frac{u(+\infty)}{\kappa} \right|^{\lambda} \int_{\omega(t)}^t g_0(s) \mathcal{K}(s) ds \quad \text{for } t \geq \tau$$

if (6.5) holds, and

$$(6.8) \quad \ln \frac{u(t)}{u(\omega(t))} \leq - \left| 1 - \frac{u(+\infty)}{\kappa} \right|^{\lambda} \int_{\omega(t)}^t g_0(s) \mathcal{K}(s) ds \quad \text{for } t \geq \tau$$

if (6.6) is fulfilled. Now both (6.7) and (6.8), in view of (6.3), results in

$$0 = \left| 1 - \frac{u(+\infty)}{\kappa} \right|^{\lambda}.$$

Consequently,  $u(+\infty) = \kappa$ . □

**Scalar differential equation without diffusion:** Consider the delay differential equation

$$(6.9) \quad u'(t) = -u(t) + G(u(t - \tau(t))) \quad \text{for } t \in \mathbb{R}$$

where

$$(6.10) \quad \tau \in C_{loc}(\mathbb{R}; (0, +\infty)), \quad \limsup_{t \rightarrow -\infty} \tau(t) < +\infty,$$

and there exists  $\kappa > 0$  such that the nonlinearity  $G$  satisfies the following conditions:

$$(6.11) \quad G \in C_{loc}(\mathbb{R}_+; \mathbb{R}_+), \quad G(0) = 0, \quad G(s) > s \quad \text{for } s \in (0, \kappa),$$

$$(6.12) \quad \lim_{s \rightarrow +\infty} \frac{q_0(s)}{s} = 0 \quad \text{where} \quad q_0(s) \stackrel{\text{def}}{=} \max \{ G(x) : x \in [0, s] \}.$$

The delay differential equation (6.9) covers, e.g., Nicholson's equation describing the blowflies population, or the Mackey-Glass equation applied to model white cell production. As an illustrative example of the function  $G$  we can consider

$$(6.13) \quad G(s) = s^p(\kappa - s) + s \quad \text{for } s \in [0, \kappa], \quad p > 0.$$

We are interested in the existence of global positive solutions to (6.9) satisfying  $u(-\infty) = 0$ . For this purpose let  $t_0 \in \mathbb{R}$  and define

$$(6.14) \quad \mu_1(t) \stackrel{\text{def}}{=} t \quad \text{for } t \in \mathbb{R}, \quad \mu_0(t) = \nu(t) \stackrel{\text{def}}{=} t - \tau(t) \quad \text{for } t \in \mathbb{R},$$

$$(6.15) \quad p_0(t) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{for } t \leq t_0, \\ 0 & \text{for } t > t_0, \end{cases} \quad p_1(t) \stackrel{\text{def}}{=} 1 \quad \text{for } t \in \mathbb{R},$$

$$(6.16) \quad h(t, x, y) \stackrel{\text{def}}{=} G(|y|) - p_0(t)y \quad \text{for } t \in \mathbb{R}, \quad x, y \in \mathbb{R},$$

$$(6.17) \quad q(t, \rho) \stackrel{\text{def}}{=} q_0(\rho) \quad \text{for } t \geq t_0, \quad \rho \in \mathbb{R}_+,$$

and consider the problem (1.5), (1.2). Then it can be easily verified that all the assumptions of Theorem 3.5 are fulfilled. Indeed, we first observe that  $p_i \in L_{loc}(\mathbb{R}; \mathbb{R}_+)$ ,  $\mu_i, \nu : \mathbb{R} \rightarrow \mathbb{R}$  are locally bounded functions ( $i = 0, 1$ ), and  $h : \mathbb{R}^3 \rightarrow \mathbb{R}$  satisfies the Carathéodory conditions mentioned in the introduction, as  $G$  is a continuous function. Moreover, the condition  $G(0) = 0$  implies that

$$h(t, 0, 0) = 0 \quad \text{for } t \in \mathbb{R},$$

and since  $G(s) \geq s$  for  $s \in [0, \kappa]$ , we have that

$$h(t, x, y) \geq 0 \quad \text{for } t \in \mathbb{R}, \quad x, y \in [0, \kappa].$$

Furthermore, (6.12) and (6.15)–(6.17) implies that

$$h(t, x, y) \operatorname{sgn} x \leq q(t, |x| + |y|) \quad \text{for } t > t_0, \quad x, y \in \mathbb{R}$$

where  $q : [t_0, +\infty) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a Carathéodory function nondecreasing in the second argument and satisfying (2.5) for every  $b > t_0$ . Finally, we also have

$$\mu_0(t) \leq t, \quad \mu_1(t) \leq t, \quad \nu(t) \leq t \quad \text{for } t \in \mathbb{R}.$$

Thus, the conditions (3.1)–(3.4) are fulfilled.

On the other hand, observe that  $p_i$  satisfy (3.6) and (3.14), as

$$\int_{\mu_1(t)}^t p_1(s) ds = 0 \quad \text{for } t \in \mathbb{R}.$$

In addition, in view of (6.10), (6.14), and (6.15) we have

$$\int_{\mu_0(t)}^t p_1(s) ds = \tau(t) \leq \sup \{ \tau(s) : s \leq t_0 \} < +\infty \quad \text{for } t \leq t_0,$$

and so also the condition (3.7) is valid. Furthermore, (6.11), (6.15), and (6.16) results in (3.15).

Thus, according to Theorem 3.5 and Remark 3.7, for every  $c \in (0, \kappa e^{-M_\tau})$  with

$$(6.18) \quad M_\tau \stackrel{\text{def}}{=} \sup \{ \tau(t) : t \leq t_0 \},$$

there exists a global solution  $u$  to the problem (1.5), (1.2) having a finite limit  $u(-\infty)$  and satisfying

$$(6.19) \quad 0 \leq u(t) \leq \kappa \quad \text{for } t \leq t_0, \quad u(t) > 0 \quad \text{for } t > t_0.$$

From (6.14)–(6.16) and (6.19) it follows that  $u$  is also a solution to (6.9). Moreover, if we put

$$h_1(x, y) \stackrel{\text{def}}{=} G(y) - y \quad \text{for } x, y \in (0, \kappa) \times (0, \kappa),$$

then (3.21) and (3.22) hold with  $g \equiv 1$ . Therefore, according to Corollary 3.5 we have  $u(-\infty) = 0$ .

Further, let

$$h_0(y) \stackrel{\text{def}}{=} \max \{G(s) - s : s \in [0, y]\} \quad \text{for } y \in \mathbb{R}_+,$$

$$g(t) = 1 \quad \text{for } t \leq t_0.$$

Then, obviously, (3.13) holds. However, (2.13) is not, generally speaking, valid, as one can check by the illustrative case (6.13). Obviously, in that case (2.13) holds if and only if  $p \geq 1$ . Consequently, Corollary 3.1 can be applied only for certain  $G$  to conclude that  $u$  is also positive on the whole real line.

However, in spite of the fact that the assumptions of Corollary 3.1, generally speaking, are not fulfilled, still we can conclude that the solution  $u$  is positive on the whole real line (i.e., also for  $p \in (0, 1)$  provided (6.13) is fulfilled). Indeed, the positivity of  $u$  is guaranteed by the following assertion.

**Lemma 6.1.** *Let (6.10) and (6.11) hold. If  $u$  is a nontrivial non-negative solution to (6.9), then  $u(t) > 0$  for  $t \in \mathbb{R}$ .*

*Proof.* Suppose on the contrary that there exists  $\eta \in \mathbb{R}$  such that  $u(\eta) = 0$ . Then we have

$$u(t) = - \int_t^\eta e^{s-t} G(u(s - \tau(s))) ds \leq 0 \quad \text{for } t \leq \eta.$$

Since  $u \geq 0$  for  $t \in \mathbb{R}$ , we can conclude that  $u(t) = 0$  for  $t \leq \eta$ . In addition, since  $u$  is a nontrivial non-negative function, there exists  $\zeta \in \mathbb{R}$  such that  $u(\zeta) > 0$ . Obviously,  $\eta < \zeta$  and without loss of generality we can assume that  $u(t) > 0$  for  $t \in (\eta, \zeta]$ .

Since  $\tau(t) > 0$  for  $t \in \mathbb{R}$  is continuous, there exists  $\varepsilon > 0$  such that  $t - \tau(t) \leq \eta$  for  $t \in [\eta, \eta + \varepsilon]$ , and hence  $u(t - \tau(t)) = 0$  for  $t \in [\eta, \eta + \varepsilon]$ . Since  $G(0) = 0$ , we have  $G(u(t - \tau(t))) = 0$  for  $t \in [\eta, \eta + \varepsilon]$ . Consequently, from (6.9) it follows that

$$u'(t) = -u(t) \quad \text{for } t \in [\eta, \eta + \varepsilon], \quad u(\eta) = 0,$$

whence we get  $u(t) = 0$  for  $t \in [\eta, \eta + \varepsilon]$ , a contradiction.  $\square$

Therefore, the above-mentioned discussion and Lemma 6.1 results in the following assertion.

**Theorem 6.2.** *Let (6.10)–(6.12) hold. Then, for each  $t_0 \in \mathbb{R}$  and  $c \in (0, \kappa e^{-M_\tau}]$  with  $M_\tau$  given by (6.18), there exists a positive global solution  $u$  to (6.9) such that*

$$u(t_0) = c, \quad u(t) \leq \kappa \quad \text{for } t \leq t_0,$$

*and there exists a limit  $u(-\infty) = 0$ .*

**Remark 6.1.** In spite of Theorem 3.5, the value  $c = \kappa e^{-M_\tau}$  is admissible in Theorem 6.2, because the function  $\tau$  is continuous and  $p_1(t) > 0$  for  $t \in \mathbb{R}$ . Consequently, a function  $\sigma$  can be directly defined as  $\sigma(t) = t - \tau(t)$  for  $t \in \mathbb{R}$  (see the proof of Theorem 3.1 for more details).

**Remark 6.2.** Note that the typical condition on  $G$ : “ $G$  is differentiable at 0” is not used in the proof of Theorem 6.2. Therefore, the results presented complete or improve the already known results.

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